Stability of Weakly Pareto-Nash Equilibria and Pareto-Nash Equilibria for Multiobjective Population Games

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Abstract

Using the method of generic continuity of set-valued mappings, this paper studies the stability of weakly Pareto-Nash and Pareto-Nash equilibria for multiobjective population games, when payoff functions are perturbed. More precisely, the paper investigates the continuity properties of the set of weakly Pareto-Nash equilibria and that of the set of Pareto-Nash equilibria under sufficiently small perturbations. Firstly, the set of weakly Pareto-Nash equilibria is proven to be upper semicontinuous and further generically continuous with payoff functions perturbed. Secondly, examples are illustrated to show that the set of Pareto-Nash equilibria is neither upper semicontinuous nor lower semicontinuous. By seeking an upper semicontinuous sub-mapping, it is shown that the set of Pareto-Nash equilibria is partly upper semicontinuous and almost lower semicontinuous.

Key words. Stability; Generic continuity; Multiobjective population game; Pareto-Nash equilibrium; Sub-mapping.

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1 Introduction

Multiobjective population games (MPGs) are the population games with vector-valued payoffs. The theory of population games [1] originates from Nash's "mass-action" interpretation of equilibrium points in his dissertation (Nash(1950b)) and his related literatures [2,3]. Population games serve as a general model for studying strategic interactions among large numbers of agents, hence they are widely applied to modelling many economic, social and technological environments in which large collections of small agents make strategically interdependent decisions, such as network congestion, public goods and externalities, cultural integration and assimilation, markets and bargaining, etc.

Recently, population games and their applications have abstracted increasing attention [4–8]. However, it is worth noting that all the payoffs in the current researches still remain in the scalar case. In other words, these models of population games were only considered as single-objective ones. As we know, in real world the criteria for choosing a strategy usually vary from population to population, even within one population, their criteria are often more than one, such as individual payoff, social position and life satisfaction, etc. Hence, a generalization of the scalar case to multiple criteria is of both theoretical and practical significance for population games.

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Weakly Pareto-Nash and Pareto-Nash equilibria, corresponding to weakly efficient and efficient solutions in vector optimization problems, have been constant topics in multiobjective games in recent years. Some scholars explored the existence of weakly Pareto-Nash and Pareto-Nash equilibria [9–11], others focused on their stability [12–14]. In essence, the stability is to study the continuity properties of the set of equilibria. This ideas has been widely used in various fields, such as optimization problems [15–18], linear systems [19], and Nash equilibrium problems [12–14, 20, 21].

In this paper, we are mainly interested in the continuity properties of the set of weakly Pareto-Nash and that of the set of Pareto-Nash equilibria for (MPGs). In particualr, about the issue of continuity, we focus on upper/lower semicontinuity of the set of weakly Pareto-Nash and that of the set of Pareto-Nash equilibria with respect to perturbations of payoff functions. Firstly, the set of weakly Pareto-Nash equilibria is proven to be upper semicontinuous further continuous on a subset of the space of continuous payoff functions equipped with uniform converge norm for (MPGs). However, the set of Pareto-Nash equilibria does not have the same satisfactory continuity properties as that of weakly Pareto-Nash equilibria for (MPGs). In fact, we illustrate two examples to show that the set-valued mapping of Pareto-Nash equilibria is neither upper semicontinuous nor lower semicontinuous. Therefore, to some extent, the continuity of the set of Pareto-Nash equilibria is to be discounted.

Inspired by [14, 16], we show that the Pareto-Nash equilibria mapping is partly upper semicontinuous although it is not upper semicontinuous by seeking an upper semicontinuous sub-mapping. To achieve this result, the continuity of the set of weighted Nash equilibria and the relationship between weighted Nash equilibria and Pareto-Nash equilibria play crucial role.

The outline of the paper is as follows: in Section 2, the concepts of weakly Pareto-Nash and Pareto-Nash equilibrium are proposed for (MPGs). And some preliminaries on set-valued mappings are reviewed. Section 3 is devoted to the stability of weakly Pareto-Nash equilibria and Pareto-Nash equilibria for (MPGs). In Section 4, a concise conclusion is made for this paper.

2 Preliminaries

Throughout this paper, for each positive integer k, denote

$$R_+^k = \{a = (a_1, \ldots, a_k) \in R^k : a_i \geq 0, j = 1, \ldots, k\},$$

and its interior

$$\text{int}R_+^k = \{a = (a_1, \ldots, a_k) \in R^k : a_i > 0, j = 1, \ldots, k\},$$

respectively. And denoted by T_+^k and $\text{int}T_+^k$ the simplex of R_+^k and its interior:

$$T_+^k = \{a = (a_1, \ldots, a_k) \in R_+^k : \sum_{j=1}^k a_j = 1\},$$

$$\text{int}T_+^k = \{a = (a_1, \ldots, a_k) \in \text{int}R_+^k : \sum_{j=1}^k a_j = 1\},$$

respectively.

Consider (MPGs), we state it as follows: let $\mathcal{P} = \{1, \ldots, P\}$ be a society consisting of $P \geq 1$ populations of agents. In each population $p \in \mathcal{P}$, there are a large but finite number of agents and they are capable of independently choosing pure strategies from a finite set $S^p = \{1, \ldots, n^p\}$. For the convenience of discussion, throughout this paper, we all assume that the mass of agents in every population is unity. Thus, for each $p \in \mathcal{P}$, denoted by $X^p = \left\{x^p = (x_1^p, \ldots, x_{n^p}^p) \in R_+^{n^p} : \sum_{i=1}^{n^p} x_i^p = 1\right\}$, the set of population states, is an $n^p - 1$ dimensional simplex, and where the nonnegative scalar x_i^p represents the share distribution of members playing strategy $i \in S^p$ in population p. Let $m = \sum_{p \in \mathcal{P}} n^p$ equal the total number of pure strategies in all populations, and $X = \prod_{p \in \mathcal{P}} X^p = \left\{x = (x^1, \ldots, x^P) \in R^m : x^p \in X^p\right\}$ denotes the set of social states, in which the element $x = (x^1, \ldots, x^P) \in X$ describes the all populations' behavior at once. And we assume that in each population $p \in \mathcal{P}$ agents all have k^p objectives whenever they play a strategy. $F_i^p : X \to R^{k^p}$ defines a vector-valued payoff function to a strategy $i \in S^p$, where the element F_{ij}^p represents the jth objective real-valued payoff to the strategy $i \in S^p$; and $F^p = (F_1^p, \ldots, F_{n^p}^p) : X \to R^{n^p k^p}$ describes population p's payoff functions for all strategies in S^p. Now let $N = \sum_{p \in \mathcal{P}} n^p k^p$, the payoff functions $F : X \to R^N$ is a map that assigns each social state a vector of payoffs, one for each criterion corresponding to each strategy in each population. Since the sets of populations and strategies are generally taken as fixed, a (MPG) is identified with its payoff functions F in the context.

The notions of weakly Pareto-Nash and Pareto-Nash equilibrium of (MPGs) are defined as follows:

Definition 2.1 *For a (MPG) F,*

(1) a social state $\bar{x} = (\bar{x}^1, \ldots, \bar{x}^P) \in X$ is called a weakly Pareto-Nash equilibrium of F if for each $p \in \mathcal{P}$,

$$\bar{x}_i^p > 0 \Rightarrow F_i^p(\bar{x}) - F_l^p(\bar{x}) \notin -intR_+^{k^p}, \ \forall i, l \in S^p.$$

(2) a social state $\bar{x} = (\bar{x}^1, \ldots, \bar{x}^P) \in X$ is called a Pareto-Nash equilibrium of F if for each $p \in \mathcal{P}$,

$$\bar{x}_i^p > 0 \Rightarrow F_i^p(\bar{x}) - F_l^p(\bar{x}) \notin -R_+^{k^p} \backslash \{0\}, \ \forall i, l \in S^p.$$

Denote the set of weakly Pareto-Nash equilibria and that of Pareto-Nash equilibria by $PE_w(F)$ and $PE(F)$, respectively.

Clearly, $PE(F) \subseteq PE_w(F)$. And if $k^p = 1$ for each $p \in \mathcal{P}$, a Pareto-Nash equilibrium and a weakly Pareto-Nash equilibrium reduce to a Nash equilibrium of population games [1].

Definition 2.2 *A social state $\bar{x} = (\bar{x}^1, \ldots, \bar{x}^P) \in X$ is called a weighted Nash equilibrium of F with respect to a given weight combination $\lambda = (\lambda^1, \cdots, \lambda^P)$ satisfying $\lambda^p \in T_+^{k^p} (\forall p \in \mathcal{P})$ if for each $p \in \mathcal{P}$,*

$$\bar{x}_i^p > 0 \Rightarrow F_{\lambda,i}^p(\bar{x}) \geq F_{\lambda,l}^p(\bar{x}), \quad \forall i, l \in S^p,$$

where $F_{\lambda,i}^p(x) = \sum_{j=1}^{k^p} \lambda_j^p F_{ij}^p(x)$ is the additive weight payoff to a strategy $i \in S^p$. And denoted by $E_\lambda(F)$ the set of all the weighted Nash equilibria of F with respect to a given weight combination λ.

Let us recall some necessary definitions and lemmas, which are helpful to main results of this paper.

Definition 2.3 *Let E and H be two topological vector spaces, Y a nonempty convex subset of E and C a closed, convex and pointed cone of H with $intC \neq \emptyset$. Let $f : Y \to H$ be a vector-valued function. f is said to be C-continuous at $y_0 \in Y$ if, for any open neighborhood V of θ in H, there exists an open neighborhood U of $y_0 \in Y$ such that, for all $y \in U$,*

$$f(y) \in f(y_0) + V + C, \tag{2.1}$$

and C-continuous on Y if it is C-continuous at any point of Y, where θ denotes the zero element of H.

Definition 2.4 *Let E and H be two topological vector spaces, Y a nonempty convex subset of E and C a closed, convex and pointed cone of H with $intC \neq \emptyset$. Let $f : Y \to H$ be a vector-valued function. f is called C-concave if for each $y_1, y_2 \in Y$ and each $t \in [0, 1]$,*

$$tf(y_1) + (1-t)f(y_2) - f\big(ty_1 + (1-t)y_2\big) \in C,$$

and C-convex if $-f$ is C-concave.

Lemma 2.5 *([12]) Let $f : X \to R^m$ be a vector-valued function, where $f = (f_l, \cdots, f_m)$. Then f is R^m_+-continuous if and only if f_i is lower semicontinuous for every $i = 1, \cdots, m$.*

Lemma 2.6 *([12]) Let X be a convex subset of a normed space and $f : X \to R^m$ be a vector-valued function, where $f = (f_l, \cdots, f_m)$. Then f is R^m_+-concave if and only if f_i is concave for every $i = 1, \cdots, m$.*

Lemma 2.7 *(Theorem 1.1 in [12]) Let E and H be two topological vector spaces, let X be a nonempty convex compact subset of E and C be a closed, convex and pointed cone of H with $intC \neq \emptyset$. Suppose that $\phi : X \times X \to H$ satisfies the following conditions:*

(i) for each fixed $y \in X, x \to \phi(x, y)$ is C-continuous;

(ii) for each fixed $x \in X, y \to \phi(x, y)$ is C-concave; and

(iii) for each $x \in X$, $\phi(x, x) \notin intC$.

Then there exists $x^* \in X$ such that $\phi(x^*, y) \notin intC$ for all $y \in X$.

Definition 2.8 *Let X and Y be two metric spaces, and let $T : Y \rightrightarrows 2^X$ be a set-valued mapping.*

(1) T is said to be upper semi-continuous at $y \in Y$ if for any open set U in X with $U \supset T(y)$, there is an open neighborhood $\mathcal{O}(y)$ of y such that $U \supset T(y')$ for each $y' \in \mathcal{O}(y)$. If T is upper semi-continuous on Y and $T(y)$ is compact for every $y \in Y$, then T is said to be an usco mapping.

(2) T is said to be lower semi-continuous at $y \in Y$ if for any open set U with $U \cap T(y) \neq \emptyset$, there is an open neighborhood $\mathcal{O}(y)$ of y such that $U \cap T(y') \neq \emptyset$ for each $y' \in \mathcal{O}(y)$.

(3) T is said to be almost lower semicontinuous at $y \in Y$ if there exists at least one $x \in T(y)$ such that, for each open neighborhood $N(x)$ of x, there exists an open neighborhood $\mathcal{O}(y)$ of y such that $\mathcal{N}(x) \cap T(y') \neq \emptyset$ for any $y' \in \mathcal{O}(y)$.

The following example provides a set-valued mapping that is almost lower semicontinuous but not lower semicontinuous.

Example 2.9 *Let $X = [0, 1]$ and define the set-valued mapping $T : X \rightrightarrows 2^X$ by*

$$T(x) = \begin{cases} \{0\}, & x \in (0,1] \ , \\ [0,1], & x = 1. \end{cases}$$

It is easy to check that T is almost lower semicontinuous but not lower semicontinuous at $x = 1$.

Lemma 2.10 *Let Y be a Hausdorff topological space, and X be compact space. If the graph(T) of the set-valued mapping $T : Y \rightrightarrows 2^X$ is closed, then T is upper semicontinuous on Y.*

Lemma 2.11 *([22]) Let X be a metric space, Y be a Baire space, and $T : Y \rightrightarrows 2^X$ be an usco mapping. Then there is a dense residual subset Q of Y such that T is a lower semicontinuous mapping at each $y \in Q$.*

Remark 2.12 *Lemma 2.11 indicates the set-valued mapping $T : Y \rightrightarrows 2^X$ is continuous on a dense residual subset Q of Y, so T is usually called generic continuity on Y.*

3 Stability of weakly Pareto-Nash and Pareto-Nash equilibrium for (MPGs)

In this section, we begin with the existence of weakly Pareto-Nash equilibrium for (MPGs).

Theorem 3.1 *Let $F : X \to R^N$ be a (MPG), where $X = \prod_{p\in\mathcal{P}} X^p$ and X^p is a simplex in R^{n^p} for each $p \in \mathcal{P}$. If F is continuous on X, then it admits at least one weakly Pareto-Nash equilibrium.*

Proof. Define a vector-valued function $\phi : X \times X \to R^k$ by

$$\phi(x, y) = \langle x - y, \widehat{F}(x)\rangle,$$

where $\widehat{F}(x) = \big(\widehat{F}^1(x), \ldots, \widehat{F}^P(x)\big)$, $k = \max_{p\in\mathcal{P}} k^p$, and $\langle \cdot, \cdot \rangle$ denotes the inner product in R^k. For any $p \in \mathcal{P}$ and $j \in S^p$, setting

$$\widehat{F}^p_j(x) = (\underbrace{F^p_j(x)}_{k^p components} \ ; \ \underbrace{F^p_{j1}(x)}_{k-k^p components}) \in R^k.$$

It is easy to check that
(i) for each fixed $y \in X, x \to \phi(x, y)$ is R^k_+−continuous (by Lemma 2.5);
(ii) for each fixed $x \in X, y \to \phi(x, y)$ is R^k_+−concave (by Lemma 2.6); and
(iii) for each $x \in X$, $\phi(x, x) = 0 \notin \mathrm{int}R^k_+$.
Therefore, by Lemma 2.7, there exists $\bar{x} \in X$ such that $\phi(\bar{x}, y) \notin \mathrm{int}R^k_+$ for all $y \in X$. i.e.,

$$\phi(\bar{x}, y) = \langle \bar{x} - y, \widehat{F}(\bar{x})\rangle \notin \mathrm{int}R^k_+, \quad \forall y \in X.$$

Now for any $p \in \mathcal{P}$, denotes $I^p(\bar{x}) = \{l \in S^p : \bar{x}_l > 0\}$, obviously, $I^p(\bar{x}) \neq \emptyset$. For each fixed $i \in I^p(\bar{x})$, for any $l \in S^p$, setting $y = (y^p, \bar{x}^{-p}) \in X$ and

$$y^p = (\bar{x}^p_1, \cdots, \bar{x}^p_{i-1}, \underbrace{0}_{i}, \bar{x}^p_{i+1}, \cdots, \bar{x}^p_{l-1}, \underbrace{\bar{x}^p_i + \bar{x}^p_l}_{l}, \bar{x}^p_{l+1}, \cdots, \bar{x}^p_{n^p}) \in X^P,$$

where $\bar{x}^{-p} = (\bar{x}^1, \cdots, \bar{x}^{p-1}, \bar{x}^{p+1}, \cdots, \bar{x}^P) \in X \backslash X^p$. Furthermore,

$$\bar{x}_i^p \big(\underbrace{F_i^p(\bar{x}) - F_l^p(\bar{x})}_{k^p components}, \underbrace{F_{i1}^p(\bar{x}) - F_{l1}^p(\bar{x})}_{k-k^p components}\big) = \phi(\bar{x}, y) \notin \mathrm{int} R_+^k.$$

Because of $\bar{x}_i^p > 0$, then

$$\big(\underbrace{F_i^p(\bar{x}) - F_l^p(\bar{x})}_{k^p components}, \underbrace{F_{i1}^p(\bar{x}) - F_{l1}^p(\bar{x})}_{k-k^p components}\big) \notin \mathrm{int} R_+^k. \tag{3.1}$$

If $F_i^p(\bar{x}) - F_l^p(\bar{x}) \in \mathrm{int} R_+^{k^p}$, then $F_{ij}^p(\bar{x}) - F_{lj}^p(\bar{x}) \in \mathrm{int} R_+^{k^p}$ for each $j = 1, 2, \cdots, k^p$ and thus

$$\big(\underbrace{F_i^p(\bar{x}) - F_l^p(\bar{x})}_{k^p components}, \underbrace{F_{i1}^p(\bar{x}) - F_{l1}^p(\bar{x})}_{k-k^p components}\big) \in \mathrm{int} R_+^k,$$

which contradicts that expression (3.1). Consequently, for all $p \in \mathcal{P}, i \in I^p(\bar{x})$, it holds true that

$$\bar{x}_i^p > 0 \Rightarrow F_i^p(\bar{x}) - F_l^p(\bar{x}) \notin \mathrm{int} R_+^{k^p}, \forall l \in S^p.$$

Hence, $\bar{x}$ is a weakly Pareto-Nash equilibrium of F. The proof is complete. ∎

Let $\mathcal{F}$ the collection of (MPGs) satisfying that: (i) X^p is a simplex in R^{n^p} for all $p \in \mathcal{P}$; (ii)$F : X = \prod_{p\in\mathcal{P}} X^p \to R^N$ is continuous. (iii)$\sum_{p\in\mathcal{P}} \max_{x\in X} \|F^p(x)\| < +\infty$.

For any $F, G \in \mathcal{F}$, define

$$\rho(F, G) = \sum_{p\in\mathcal{P}} \max_{x\in X} \|F^p(x) - G^p(x)\|.$$

Clearly, ρ is a metric on $\mathcal{F}$. Indeed, $(\mathcal{F}, \rho)$ is complete. From Theorem 3.1, $PE_w(F) \neq \emptyset$ for each $F \in \mathcal{F}$. And PE_w, PE are both set-valued mappings from $\mathcal{F}$ to X, which are called weakly Pareto-Nash and Pareto-Nash equilibrium mapping in the remainder of this paper, respectively.

Definition 3.2 *For each $F \in \mathcal{F}$, let $x \in PE_w(F)$ (resp. $x \in PE(F)$). Then x is said to be an essential weakly Pareto-Nash (resp. essential Pareto-Nash) equilibrium of F provided that for any open set $U \supset x$, there exists an open neighborhood $\mathcal{N}(F)$ of $F \in \mathcal{F}$ such that $U \cap PE_w(F') \neq \emptyset$ (resp. $U \cap PE(F') \neq \emptyset$) for any $F' \in \mathcal{N}(F)$. If a subset of $PE_w(F)$ (resp. $PE(F)$) has the same property, we call it the essential set.*

Remark 3.3 *x is an essential weakly Pareto-Nash (resp. essential Pareto-Nash) equilibrium of $F \in \mathcal{F}$, namely, the mapping PE_w (resp. PE) is lower semicontinuous at F.*

Theorem 3.4 *The set-valued mapping $PE_w : \mathcal{F} \rightrightarrows 2^X$ is an usco mapping on $\mathcal{F}$.*

Proof. Since $X = \prod_{p\in\mathcal{P}} X^p$ is compact, from Lemma 2.10, it suffices to verify that graph(PE_w) of the set-valued mapping PE_w is closed, where graph$(PE_w) = \{(F, x) \in \mathcal{F} \times X : x \in PE_w(F)\}$. Let $(F^n, x^n) \in$ graph(PE_w). Assume that $(F^n, x^n) \to (\bar{F}, \bar{x}) \in \mathcal{F} \times X$, then $F^n \in \mathcal{F}, x^n \in PE_w(F^n)$. It needs to prove $\bar{x} \in PE_w(\bar{F})$. Argue by contradiction. Suppose that $\bar{x} \notin PE_w(\bar{F})$, then there are some $p_0 \in \mathcal{P}$ and $i_0, l_0 \in S^{p_0}$, though $\bar{x}_{i_0}^{p_0} > 0$,

$$\bar{F}_{i_0}^{p_0}(\bar{x}) - \bar{F}_{l_0}^{p_0}(\bar{x}) \in -\mathrm{int} R_+^{k^{p_0}}.$$

As $(F^n, x^n) \to (\bar{F}, \bar{x})$, then $F^n \to \bar{F}, x^n \to \bar{x}$, there exists $N_1 \in \mathbb{N}$, such that $F^n(\bar{x}) \to \bar{F}(\bar{x})$ and $(x^n)_{i_0}^{p_0} \to \bar{x}_{i_0}^{p_0}$ with $(x^n)_{i_0}^{p_0} > 0$ whenever $n > N_1$. Further by combining the continuity of F^n on X with the fact $F^n \to \bar{F}, x^n \to \bar{x}$, then there exists $N_2 \in \mathbb{N}$ and $N_2 > N_1$, such that

$$(F^n)_{i_0}^{p_0}(x^n) - (F^n)_{l_0}^{p_0}(x^n) \in -\mathrm{int}R_+^{k^{p_0}},$$

whenever $n > N_2$.

In a word, although $(x^n)_{i_0}^{p_0} > 0$, $(F^n)_{i_0}^{p_0}(x^n) - (F^n)_{l_0}^{p_0}(x^n) \in -\mathrm{int}R_+^{k^{p_0}}$ whenever $n > N_2$. By Definition 2.1, thus $x^n \notin PE_w(F^n)$, which contradicts the fact $x^n \in PE_w(F^n)$. Therefore, $\bar{x} \in PE_w(\bar{F})$, and hence graph$(PE_w)$ is closed. Following from Lemma 2.10, then PE_w is upper semicontinuous on $\mathcal{F}$. Observing that $X = \prod_{p\in\mathcal{P}} X^p$ is compact, so graph(PE_w) is also compact for each $F \in \mathcal{F}$. The proof is complete. ∎

From Theorem 3.4 and Lemma 2.11, it is easy to obtain the generic continuity of the set of weakly Pateto-Nash equilibrium for (MPGs) as follows:

Theorem 3.5 *There exists a dense residual subset Q of $\mathcal{F}$ such that the weakly Pareto-Nash equilibrium mapping $PE_w : \mathcal{F} \rightrightarrows 2^X$ is lower semicontinuous furhter continuous on Q. That is, every weakly Pareto-Nash equilibrium of each $F \in Q$ is essential.*

Proof. From Theorem 3.4 and Lemma 2.11, there exists a dense residual subset Q of $\mathcal{F}$ such that PE_w is lower semicontinuous further continuous at each $F \in Q$. Let $x^* \in PE_w(F)$ for $F \in Q$. Since PE_w is lower semicontinuous at F, then x^* is an essential weakly Pareto-Nash equilibrium from Remark 3.3 and Definition 3.2. The proof is complete. ∎

One can observe that such generic continuity on the set of weakly Pareto-Nash equilibria is due to its upper semicontinuity property. On involving the set of Pareto-Nash equilibria of (MPGs), however, it is not a satisfactory one. The following example shows this point.

Example 3.6 *Let F be a bi-objective population game played by two unit mass populations with two strategies for each ($n^1 = n^2 = 2$). Let the corresponding sets of population states be $Y = \{y = (y_1, y_2) \in R_+^2 : y_1 + y_2 = 1\}, Z = \{z = (z_1, z_2) \in R_+^2 : z_1 + z_2 = 1\}$, respectively. Let $X = Y \times Z = \{x = (y, z) : y \in Y, z \in Z\}$ and b be a real constant. Let $F(x) = (F^1(x), F^2(x))$, where for population 1, $F^1(x) = (F_1^1(x), F_2^1(x))$,*

$$\begin{aligned} F_1^1(x) &= \big(F_{11}^1(x), F_{12}^1(x)\big) = (y_1, y_2), \\ F_2^1(x) &= \big(F_{21}^1(x), F_{22}^1(x)\big) = (1 + y_1, 1 + y_2). \end{aligned}$$

And for population 2, $F^2(x) = (F_1^2(x), F_2^2(x))$,

$$\begin{aligned} F_1^2(x) &= \big(F_{11}^2(x), F_{12}^2(x)\big) = (z_1, b), \\ F_2^2(x) &= \big(F_{21}^2(x), F_{22}^2(x)\big) = (1 + z_1, b). \end{aligned}$$

It is easy to see that the payoff to the 2nd strategy dominates that of the 1st strategy in each population, thus $PE(F)$ contains only one state $\bar{x} = \{(0, 1), (0, 1)\}$, i.e., $PE(F) = \{\bar{x}\}$.

Let $F^n(x) = \big((F^n)^1(x), (F^n)^2(x)\big)$, where $(F^n)^1(x) = F^1(x)$;

$$\begin{aligned} (F^n)_1^2(x) &= \big((F^n)_{11}^2(x), (F^n)_{12}^2(x)\big) = (z_1, b + 1/n), \\ (F^n)_2^2(x) &= \big((F^n)_{21}^2(x), (F^n)_{22}^2(x)\big) = (1 + z_1, b). \end{aligned}$$

Clearly, $F^n \to F (n \to +\infty)$, and we find that the Pareto-Nash equilibria of F^n is $PE(F^n) = \{(0,1)\} \times Z$ since for all $x \in X$, $(F^n)^1(x) = F^1(x)$; yet

$$\begin{array}{rcl}(F^n)^2_{11}(x) = z_1 & < & (F^n)^2_{21}(x) = 1 + z_1, \\ (F^n)^2_{12}(x) = b + 1/n & > & (F^n)^2_{22}(x) = b.\end{array}$$

It is easy to check that all the other states $x = (y, z) \in (Y \backslash \{(0,1)\}) \times Z$ are not Pareto-Nash equilibrium of F^n. Therefore, $PE(F^n) = \{(0,1)\} \times Z$ is the Pareto-Nash equilibria of F^n.

So the Pareto-Nash equilibrium mapping PE is not upper semicontinuous at F, because however close F^n approaches F, the Pareto-Nash equilibria of F^n cannot be covered in any small neighborhood of $\bar{x} = \{(0,1), (0,1)\}$, which is the unique Pareto-Nash equilibrium of F. However, it is not difficult to examine that PE_w is upper semicontinuous at F, because all $x = (y, z) \in \{(0,1)\} \times Z$ are weakly Pareto-Nash equilibria of F and of F^n as well.

To obtain the stability of Pareto-Nash equilibria for (MPGs), we define a sub-mapping and partly upper semicontinuity of the Pareto-Nash equilibrium mapping PE below:

Definition 3.7 *A mapping $E_0 : \mathcal{F} \rightrightarrows 2^X$ is said to be a sub-mapping of the Pareto-Nash equilibrium mapping PE if $E_0(F) \subset PE(F)$ holds for each $F \in \mathcal{F}$. Furthermore, if the sub-mapping $E_0 : \mathcal{F} \rightrightarrows 2^X$ is upper semicontinuous, then the Pareto-Nash equilibrium mapping PE is said to be partly upper semicontinuous.*

Example 3.8 *Consider a bi-objective single-population game $F = (F_1, F_2)$ with two strategies. The state set is denoted by $X = \{x = (x_1, x_2) \in R^2_+ : x_1 + x_2 = 1\}$.*

For each $x = (x_1, x_2) \in X$,

$$\begin{aligned} F_1(x) &= (F_{11}(x), F_{12}(x)) = (x_1, 2), \\ F_2(x) &= (F_{21}(x), F_{22}(x)) = (1 + x_1, x_2). \end{aligned}$$

Clearly, each $x = (x_1, x_2) \in X$ is a Pareto-Nash equilibrium of F, i.e., $PE(F) = X$.

For a given weight combination $\lambda = (\lambda_1, \lambda_2) = (1/3, 2/3) \in intT^2_+$, the resulting additive weight payoffs to strategy 1 and 2:

$$\begin{aligned} F_{\lambda,1}(x) &= \lambda_1 F_{11}(x) + \lambda_2 F_{12}(x) = x_1/3 + 4/3, \\ F_{\lambda,2}(x) &= \lambda_1 F_{21}(x) + \lambda_2 F_{22}(x) = (1 + x_1)/3 + 2x_2/3 = -x_1/3 + 1, \end{aligned}$$

respectively. By Definition 2.2, obviously, F has only one weighted Nash equilibrium $\{(1,0)\} \in X$ with respect to the given weight combination $\lambda = (\lambda_1, \lambda_2) = (1/3, 2/3) \in intT^2_+$, since $F_{\lambda,1}(x) > F_{\lambda,2}(x)$ holds for any $x = (x_1, x_2) \in X$ no matter what value $x_2 > 0$ takes except for the state $\{(1,0)\} \in X$. Consequently, $E_\lambda(F) = \{(1,0)\} \subset X = PE(F)$. By Definition 3.7, the weighted Nash equilibrium mapping is a sub-mapping of the Pareto-Nash equilibrium mapping PE for F.

Now, by seeking an upper semicontinous sub-mapping, we obtain the following partly upper semicontinuity result of the Pareto-Nash equilibrium mapping PE.

Theorem 3.9 *There exists an upper semicontinuous sub-mapping of PE, i.e., $E_0 : \mathcal{F} \rightrightarrows 2^X$ such that $E_0(F) \subset PE(F)$ for each $F \in \mathcal{F}$ and upper semicontinuous on $\mathcal{F}$. That is, PE is partly upper semicontinuous.*

Proof. Given a weight combiniation $\lambda=(\lambda^1,\cdots,\lambda^P)$ with $\lambda^p\in \text{int}T_+^{k^p}(\forall p\in\mathcal{P})$, for each $F\in\mathcal{F}$, since the additive weight payoff $F_{\lambda,i}^p(x)$ to the strategy $i\in S^p$ is continuous on X for each population $p\in\mathcal{P}$, then there exists weight Nash equilibria for each $F\in\mathcal{F}$ from the Theorem 2.1.1 ([1]), i.e., $E_\lambda(F)\neq\emptyset$ and E_λ is a set-valued mapping from $\mathcal{F}$ to X.

Setting $E_0=E_\lambda:\mathcal{F}\rightrightarrows 2^X$. Next, it is proven that the mapping $E_\lambda:\mathcal{F}\rightrightarrows 2^X$ usco.

Let $(F^n,x^n)\in\text{graph}(E_\lambda)$. Assume that $(F^n,x^n)\to(\bar{F},\bar{x})\in\mathcal{F}\times X$, then $F^n\in\mathcal{F},x^n\in E_\lambda(F^n)$. Suppose that $\bar{x}\notin E_\lambda(\bar{F})$, then there exist some $p_0\in\mathcal{P}$ and $i_0,l_0\in S^{p_0}$, $\bar{x}_{i_0}^{p_0}>0$, however,

$$\bar{F}_{\lambda,i_0}^{p_0}(\bar{x})<\bar{F}_{\lambda,l_0}^{p_0}(\bar{x}).$$

As $F_\lambda^n\to\bar{F}_\lambda$ due to $F^n\to\bar{F}$, further,

$$(F^n)_{\lambda,i_0}^{p_0}(\bar{x})<(F^n)_{\lambda,l_0}^{p_0}(\bar{x}).$$

Because $(F^n)_{\lambda,l}^{p_0}(x)$ is continuous on X for any $l\in S^p$ and $x^n\to\bar{x}$, then $(x^n)_{i_0}^{p_0}\to\bar{x}_{i_0}^{p_0}$ satisfying $(x^n)_{i_0}^{p_0}>0$, and

$$(F^n)_{\lambda,i_0}^{p_0}(x^n)<(F^n)_{\lambda,l_0}^{p_0}(x^n),$$

whenever n is sufficiently large.

In a word, $(x^n)_{i_0}^{p_0}>0$, yet $(F^n)_{\lambda,i_0}^{p_0}(x^n)<(F^n)_{\lambda,l_0}^{p_0}(x^n)$ whenever n is sufficiently large. By Definition 2.2, thus $x^n\notin E_\lambda(F^n)$, which contradicts the assumption $x^n\in E_\lambda(F^n)$. Therefore, $\bar{x}\in E_\lambda(\bar{F})$, namely, graph$(E_\lambda)$ is closed. From Lemma 2.10, then E_λ is upper semicontinuous on $\mathcal{F}$. Furthermore, notice that $X=\prod_{p\in\mathcal{P}}X^p$ is compact, so $E_\lambda(F)$ is also compact for each $F\in\mathcal{F}$. Consequently, E_λ is an usco mapping on $\mathcal{F}$.

Besides, it holds ture that $E_\lambda(F)\subset PE(F)$ for $\lambda=(\lambda^1,\cdots,\lambda^P)$ with $\lambda^p\in\text{int}T_+^{k^p},\forall p\in\mathcal{P}$. If not, there is one point $\tilde{x}\in E_\lambda(F)$, nevertheless, $\tilde{x}\notin PE(F)$. Then there are some $p_0\in\mathcal{P}$ and $i_0,l_0\in S^{p_0}$, $\tilde{x}_{i_0}^{p_0}>0$, but $F_{i_0}^{p_0}(\tilde{x})-F_{l_0}^{p_0}(\tilde{x})\in -R_+^{k^{p_0}}\backslash\{0\}$, i.e.,

$$\begin{aligned}F_{l_0j}^{p_0}(\tilde{x})&\geq F_{i_0j}^{p_0}(\tilde{x}),\ \forall j\in\{1,2,\cdots,k^{p_0}\},\text{and}\\ F_{l_0j}^{p_0}(\tilde{x})&>F_{i_0j}^{p_0}(\tilde{x}),\text{for some } j\in\{1,2,\cdots,k^{p_0}\}.\end{aligned}$$

Since $\lambda^{p_0}\in\text{int}T_+^{k^{p_0}}$ within $\lambda=(\lambda^1,\cdots,\lambda^P)$, it immediately follows that $F_{\lambda,l_0}^{p_0}(\tilde{x})>F_{\lambda,i_0}^{p_0}(\tilde{x})$. To sum up, $\tilde{x}_{i_0}^{p_0}>0$, however, $F_{\lambda,l_0}^{p_0}(\tilde{x})>F_{\lambda,i_0}^{p_0}(\tilde{x})$. From Definition 2.2, this means that $\tilde{x}\notin E_\lambda(F)$, which is a contradiction. The proof is complete. ∎

Remark 3.10 *From the proof of Theorem 3.9, along with Example 3.8, it is known that: (1) for each $F\in\mathcal{F}$, $E_\lambda(F)\neq\emptyset$ and $E_\lambda(F)\subset PE(F)$ for a given $\lambda=(\lambda^1,\cdots,\lambda^P)$ with $\lambda^p\in intT_+^{k^p},\forall p\in\mathcal{P}$;*

(2) The set of weighted Nash equilibria is usco on $\mathcal{F}$. From Lemma 2.11, there is a dense residual subset Q of $\mathcal{F}$ such that the set of weighted Nash equilibria is lower semicontinuous further continuous on Q.

(3) Furthermore, $E_\lambda(F)$ is an essential set of $PE(F)$.

The following special case shows that the Pareto-Nash equilibrium mapping PE is not lower semicontinuous on $\mathcal{F}$.

Example 3.11 *Let us consider a bi-objective single-population game $F = (F_1, F_2)$ with two strategies. Then the state set is denoted by $X = \{x = (x_1, x_2) \in R^2_+ : x_1 + x_2 = 1\}$, the simplex in R^2. Let a, b be real constants.*

For each $x = (x_1, x_2) \in X$,

$$F_1(x) = (F_{11}(x), F_{12}(x)) = (a, b),$$
$$F_2(x) = (F_{21}(x), F_{22}(x)) = (a, b).$$

Clearly, each $x = (x_1, x_2) \in X$ is a Pareto-Nash equilibrium of F, i.e., $PE(F) = X$.

Let $F^n = \big((F^n)_1, (F^n)_2\big)$ be an approximating sequence of F in which for each $x = (x^1, x^2) \in X$ we have

$$(F^n)_1(x) = \big((F^n)_{11}(x), (F^n)_{12}(x)\big) = (a, b),$$
$$(F^n)_2(x) = \big((F^n)_{21}(x), (F^n)_{22}(x)\big) = (a + 1/n, b + 1/n).$$

Obviously, the perturbed bi-objective population game F^n admits only one Pareto-Nash equilibrium $\{(0, 1)\} \in X$, namely, $PE(F^n) = \{(0, 1)\}$. Nevertheless, for a special Pareto-Nash equilibrium $\{(1, 0)\} \in PE(F)$, we can choose a small enough neighborhood $\mathcal{N}(1, 0)$; no matter how close F^n is to F, $\{(0, 1)\} \cap \mathcal{N}(1, 0) = \emptyset$. Therefore, PE is not lower semicontinuous at F. Meanwhile, it is easy to check that PE_w is also not lower semicontinuous at F, too, as $PE_w(F) = X$ and $PE_w(F^n) = \{(0, 1)\}$.

Theorem 3.12 *There exists a dense residual subset Q of $\mathcal{F}$ such that the Pareto-Nash equilibrium mapping PE is almost lower semicontinuous at each $F \in Q$.*

Proof. From Lemma 2.11 and Remark 3.10(2), there exists a dense residual subset Q of $\mathcal{F}$ such that the sub-mapping $E_0 = E_\lambda$ is lower semicontinuous at each $F \in Q$, where $\lambda = (\lambda^1, \cdots, \lambda^P)$ satisfying $\lambda^p \in \text{int}T^{k^p}_+, \forall p \in \mathcal{P}$. Let $x^* \in E_\lambda(F)$ for $F \in Q$. Then for any open neighborhood $\mathcal{O}(x^*)$ of x^*, there exists an open neighborhood $\mathcal{N}(F)$ of F such that $\mathcal{O}(x^*) \cap E_\lambda(F') \neq \emptyset$ for all $F' \in \mathcal{N}(F)$. Since $E_0(F) \subset PE(F)$, it also holds that $\mathcal{O}(x^*) \cap PE(F) \neq \emptyset$. Therefore, PE is almost lower semicontinuous on Q by Definition 2.8(3). The proof is complete. ∎

Remark 3.13 *(1) There exists at least one essential Pareto-Nash equilibrium $x^* \in PE(F)$ for most $F \in \mathcal{F}$ by the proof of Theorem 3.12.*

(2) Combining the proof of Theorem 3.12 with Example 3.6, each $x \in PE(F)$ being an essential Pareto-Nash equilibrium is a necessary but not sufficient condition for the continuity of PE at $F \in \mathcal{F}$.

4 Conclusion

In this paper, we have proven some stability results for weakly Pareto-Nash and Pareto-Nash equilibria of (MPGs) with payoff functions perturbed. Here, the stability is dependent on the semicontinuity property of the set-valued mapping $PE_w(F)$ (resp. $PE(F)$), which associates to a continuous multiobjective population game F. The weakly Pareto-Nash equilibrium mapping PE_w is upper semicontinuous. This leads to the generic continuity of weakly Pareto-Nash equilibrium, that is, each weakly Pareto-Nash equilibrium is stable for most continuous (MPGs) in the sense of Baire category. However, the problem is nontrivial as the Pareto-Nash equilibrium mapping $PE(F)$ is generally neither upper semicontinuous nor lower semicontinuous (see Example 3.6 and 3.11). We prove the partly upper semicontinuity of $PE(F)$ by seeking an upper semicontinuous sub-mapping. Based on this fact, along with the generic continuity of the set of weighted Nash equilibria (Remark 3.10(2)), it is furthermore shown that $PE(F)$ is almost lower semicontinuous for most (PMGs). Therefore, most (PMGs) have at least one stable Pareto-Nash equilibrium in the sense of Baire category. Our work extends population games with single objective ([1]) to multiobjective cases. And our results are new for population games.

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Conflict of Interest

The authors declare that they have no conflict of interest.

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