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# SOLVABILITY OF VECTOR KY FAN INEQUALITIES WITH APPLICATIONS*

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**Abstract** This paper aims to study the solvability of vector Ky Fan inequalities and the compactness of its solution sets. For vector-valued functions with the cone semicontinuity and the cone quasiconvexity in infinite dimensional spaces, the authors prove some existence results of the solutions and the compactness of the solution sets. Especially, some results for the vector Ky Fan inequalities on noncompact sets are built and the compactness of its solution sets are also discussed. As applications, some existence theorems of the solutions of vector variational inequalities are obtained.

**Keywords** Compactness, noncompact set, solvability, vector Ky Fan inequalities, vector variational inequality.

## 1 Introduction and Preliminaries

In [1], Fan introduced an important inequality which now has been called Ky Fan inequality. Till now, such an inequality plays an important role in variational analysis (it embraces Stampacchia and Minty inequalities) and hence in optimization. Ky Fan inequality provides a unified frame for some mathematical problems such as mathematical programming, saddle point, fixed point, variational inequality, complementarity problem, economical equilibrium, noncooperative game. In 1994, Blum and Oettli[2] proposed the term "equilibrium problem". Almost at the same time, Tan, Yu, and Yuan[3] named the solutions of the Ky Fan inequality "Ky Fan's points". From around 1997, lots of researchers began at about the same time to study the vector Ky Fan inequalities (some authors called vector equilibrium problems). Among the researchers are Ansari[4], Bianchi, Hadjisawas, and Schaible[5], Oettli[6], Yang and Goh[7], Chen and Goh[8], Giannessi[9], and so on. Before that time, they had studied vector optimization

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problems, vector complementarity problems and vector variational inequalities for more than ten years, see, for example, [10–12]. Vector Ky Fan inequalities are natural generalizations of the Ky Fan inequality to vector-valued functions, and contain many mathematical problems such as vector variational inequality, vector optimization, vector complementarity, multiobjective game as special cases. The study of vector Ky Fan inequalities has recently been a rapidly growing area of research, see, for example, [9, 13, 14], and the references therein. In this paper, we will study the solvability of vector Ky Fan inequalities and the compactness of its solution sets. Especially, we will built some results for the vector Ky Fan inequalities on noncompact sets and discuss the compactness of its solution sets.

Let $C$ be a nonempty subset of a Hausdorff topological vector space $H$. $C$ is called a cone if $\lambda x \in C$ for any $x \in C$ and any $\lambda > 0$. A cone $C$ is closed if $C$ is a closed set, convex if $C$ is a convex set, and pointed if $C \cap (-C) = \{0\}$. We know that there must be $\lambda C = C$ for any $\lambda > 0$ if $C$ is a cone, $0 \in C$ if $C$ is a closed cone, and $C + C = C$ if $C$ is a convex cone.

In the following context, unless special remark, $H$ denotes a Hausdorff topological vector spaces and $C$ denotes a nonempty, closed, convex, and pointed cone in $H$ with $\mathrm{int}C \neq \emptyset$.

Let $X$ be a nonempty set, and $\phi : X \times X \to H$ be a vector-valued function. This paper deals with the vector Ky Fan inequality (for short, VKFI) consisting of finding a point $x^* \in X$ such that

$$\phi(x^*, y) \notin -\mathrm{int}C, \quad \forall y \in X.$$

If such a point $x^*$ exists, we call it a solution of the VKFI.

When $H = R$ and $C = (-\infty, 0]$, the VKFI becomes the Ky Fan inequality (see [1, 3]), which is to find $x^* \in X$ such that

$$\phi(x^*, y) \leq 0, \quad \forall y \in X.$$

Let us recall some definitions and lemmas about vector-valued functions. The following three definitions can be found in [12] of Luc or [13] of Chen, et al.

**Definition 1.1** Let $X$ be a nonempty subset of a Hausdorff topological space $E$ and $f : X \to H$ be a vector-valued function. Then $f$ is said to be $C$-upper semi-continuous at $x \in X$ if for any open neighborhood $V$ of 0 in $H$, there exists an open neighborhood $U$ of $x$ in $X$ such that for any $x' \in U$,

$$f(x') \in f(x) + V - C;$$

$f$ is said to be $C$-upper semi-continuous on $X$ if $f$ is $C$-upper semi-continuous at each $x \in X$; and $f$ is said to be $C$-lower semi-continuous on $X$ if $-f$ is $C$-upper semi-continuous on $X$.

**Definition 1.2** Let $X$ be a nonempty and convex subset of a Hausdorff topological vector space $E$, and $f : X \to H$ be a vector-valued function. Then $f$ is said to be $C$-convex on $X$ if for any $x_1, x_2 \in X$ and any $\lambda \in (0, 1)$,

$$\lambda f(x_1) + (1 - \lambda)f(x_2) - f(\lambda x_1 + (1 - \lambda)x_2) \in C,$$

and $f$ is said to be $C$-concave on $X$ if $-f$ is $C$-convex on $X$.

**Definition 1.3** Let $X$ be a nonempty and convex subset of a Hausdorff topological vector space $E$ and $f : X \to H$ be a vector-valued function. Then $f$ is said to be $C$-quasiconvex on $X$ if for any $z \in H$, any $x_1, x_2 \in X$, and any $\lambda \in [0,1]$,

$$f(x_1) \in z - C, \quad f(x_2) \in z - C \Rightarrow f(\lambda x_1 + (1-\lambda)x_2) \in z - C,$$

and $f$ is said to be $C$- quasiconcave on $X$ if $-f$ is $C$- quasiconvex on $X$.

**Lemma 1.4** (see [15]) *It holds that* $\mathrm{int}C + C \subset \mathrm{int}C$ *and* $\mathrm{int}C$ *is a convex cone.*

**Lemma 1.5** *Let $X$ be a nonempty subset of a Hausdorff topological vector space $E$ and $f : X \to H$ be a vector-valued function. Then*

1) *$f$ is $C$-upper semi-continuous on $X$ if and only if for any $z \in H$, the set $L(z) = \{x \in X : f(x) \in z - \mathrm{int}C\}$ is open in $X$;*

2) *$f$ is $C$-lower semi-continuous on $X$ if and only if, for any $z \in H$, the set $L'(z) = \{x \in X : f(x) \in z + \mathrm{int}C\}$ is open in $X$.*

*Proof* We only prove 1).

$\Rightarrow$ Suppose $f$ is $C$-upper semi-continuous on $X$. Let $x \in L(z)$, then $f(x) \in z-$ $\mathrm{int}C$. Since $\mathrm{int}C$ is open, there exists an open neighborhood $V$ of 0 in $H$ such that $f(x) + V \subset z-$ $\mathrm{int}C$. Since $f$ is $C$-upper semi-continuous at $x$, for the open neighborhood $V$ of 0 in $H$, there exists an open neighborhood $U$ of $x$ in $X$ such that for any $x' \in U$,

$$f(x') \in f(x) + V - C \subset z - \mathrm{int}C - C.$$

By Lemma 1.4, $\mathrm{int}C + C \subset \mathrm{int}C$. Hence $f(x') \in z-$ $\mathrm{int}C$ for any $x' \in U$, which shows $U \subset L(z)$. Therefore $L(z)$ must be open.

$\Leftarrow$ Suppose $L(z)$ is open in $X$ for any $z \in H$. Let $V$ be an open neighborhood of 0 in $H$. By Lemma 1.4, $\mathrm{int}C$ is a convex cone, which ensures $V \cap$ $\mathrm{int}C \neq \emptyset$. For any $x \in X$, take $z \in f(x) + V \cap$ $\mathrm{int}C$, then $f(x) \in z - \mathrm{int}C$, i.e., $x \in L(z)$. Since $L(z)$ is open, there exists an open neighborhood $U$ of $x$ in $X$ such that $U \subset L(z)$. Hence, for any $x' \in U$, we have

$$f(x') \in z - \mathrm{int}C \subset f(x) + V \cap \mathrm{int}C - \mathrm{int}C \subset f(x) + V - C.$$

Thus we have shown that $f$ is $C$-upper semi-continuous at $x$. It follows from the arbitrariness of $x \in X$ that $f$ is $C$-upper semi-continuous on $X$. ∎

**Corollary 1.6** *Let $X$ be a nonempty subset of a Hausdorff topological vector space $E$ and $f : X \to H$ be a vector-valued function. Then*

1) *$f$ is $C$-upper semi-continuous on $X$ if and only if for any $z \in H$, the set $\{x \in X : f(x) \notin z-$ $\mathrm{int}C\}$ is closed in $X$;*

2) *$f$ is $C$-lower semi-continuous on $X$ if and only if for any $z \in H$, the set $\{x \in X : f(x) \notin z+$ $\mathrm{int}C\}$ is closed in $X$.*

**Lemma 1.7** *Let $X$ be a nonempty and convex subset of a Hausdorff topological vector space $E$ and $f : X \to H$ be a vector-valued function. Then*

1) *$f$ is $C$-quasiconvex if and only if for any $z \in H$, the set $L(z) = \{x \in X : f(x) \in z - \mathrm{int}C\}$ is convex;*

2) *$f$ is $C$-quasiconcave if and only if for any $z \in H$, the set $L'(z) = \{x \in X : f(x) \in z + \mathrm{int}C\}$ is convex.*

*Proof* We only prove 1).

$\Rightarrow$ Suppose $f$ is $C$-quasiconvex. Let $x_1, x_2 \in L(z)$, then $f(x_1) \in z - \mathrm{int}C$ and $f(x_2) \in z - \mathrm{int}C$. Since $-\mathrm{int}C$ is open, there exists an open neighborhood $V$ of 0 in $H$ such that $f(x_1) + V \subset z - C$ and $f(x_2) + V \subset z - C$. Take $y \in V \cap \mathrm{int}C$, then

$$f(x_1) \in z - y - C, \ f(x_2) \in z - y - C, \quad \text{and} \quad z - y \in H.$$

Since $f$ is $C$-quasiconvex, for any $\lambda \in [0, 1]$, we have

$$f(\lambda x_1 + (1 - \lambda)x_2) \in z - y - C \subset z - \mathrm{int}C - C.$$

By Lemma 1.4, we get

$$f(\lambda x_1 + (1 - \lambda)x_2) \in z - \mathrm{int}C - C \in z - \mathrm{int}C.$$

Therefore, $\lambda x_1 + (1 - \lambda)x_2 \in L(z)$ and $L(z)$ must be convex.

$\Leftarrow$ Suppose $L(z)$ is convex for any $z \in H$. For a given $z \in H$, let $x_1, x_2 \in X$ with $f(x_1) \in z - C$ and $f(x_2) \in z - C$. By Lemma 1.4, $\mathrm{int}C$ is a cone. Take $y \in \mathrm{int}C$, then we have $ty \in \mathrm{int}C$ for any $t > 0$. Thus, we get

$$f(x_1) - ty \in z - C - \mathrm{int}C \text{ and } f(x_2) - ty \in z - C - \mathrm{int}C.$$

By Lemma 1.4 again,

$$f(x_1) - ty \in z - \mathrm{int}C \quad \text{and} \quad f(x_2) - ty \in z - \mathrm{int}C,$$

i.e.,

$$f(x_1) \in (z + ty) - \mathrm{int}C \quad \text{and} \quad f(x_2) \in (z + ty) - \mathrm{int}C.$$

Since $z + ty \in H$, we have $x_1, x_2 \in L(z + ty)$. It follows from the convexity of the set $L(z + ty)$ that $\lambda x_1 + (1 - \lambda x_2) \in L(z + ty)$ for any $\lambda \in [0, 1]$. Consequently, we have

$$f(\lambda x_1 + (1 - \lambda x_2)) \in z + ty - \mathrm{int}C,$$

i.e.,

$$f(\lambda x_1 + (1 - \lambda x_2)) - ty \in z - \mathrm{int}C.$$

Let $t \to 0^+$, then by the closeness of $C$, we have $f(\lambda x_1 + (1 - \lambda x_2)) \in z - C$. Thus we have proved that $f$ is $C$-quasiconvex. ∎

**Remark 1.8** Lemma 1.7 is different from Proposition 6.3 in [12] or Proposition 1.68 in [13].

**Remark 1.9** Let $X$ be a nonempty and convex subset of a Hausdorff topological vector space $E$ and $f : X \to H$ be a vector-valued function. By Lemma 4.5.1 in [16], if $f$ is $C$-convex then it must be $C$-quasiconvex.

The following well-known KKMF lemma is an important generalization of KKM theorem to the infinite dimensional space by Ky Fan[17].

**Lemma 1.10** (KKMF) *Let $X$ be a nonempty subset of a Hausdorff topological vector space $E$. Suppose $F : X \to 2^E$ is a set-valued mapping such that $F(x)$ is a nonempty closed subset of $E$ for each $x \in X$, $F(x_0)$ is compact for at least one $x_0 \in X$, and $\mathrm{co}\{x_1, x_2, \cdots, x_n\} \subset \cup_{i=1}^n F(x_i)$ for any finite subset $\{x_1, x_2, \cdots, x_n\}$ of $X$, then $\cap_{x\in X} F(x) \neq \emptyset$.*

## 2 Solvability of VKFIs and Compactness of Its Solution Sets

**Theorem 2.1** *Let $X$ be a nonempty, convex, and compact subset of a Hausdorff topological vector space $E$. If $\psi, \phi : X \times X \to H$ satisfy the following conditions:*

(i) *for each $y \in X$, $\psi(y, y) \notin -\mathrm{int}C$;*

(ii) *for each $x \in X$, $y \to \psi(x, y)$ is $C$-quasiconvex;*

(iii) *for each $y \in X$, $x \to \phi(x, y)$ is $C$-upper semi-continuous; and*

(iv) *for each $x, y \in X$, $\psi(x, y) - \phi(x, y) \in -C$.*

*Then*

1) *there exists $x^* \in X$ such that $\phi(x^*, y) \notin -\mathrm{int}C$ for all $y \in X$; and*

2) *the solution set $S := \{x \in X : \phi(x, y) \notin -\mathrm{int}C, \ \forall y \in X\}$ is compact.*

*Proof* 1) Let a set-valued mapping $F : X \to 2^X$ be defined by

$$F(y) = \{x \in X : \phi(x, y) \notin -\mathrm{int}C\}, \ \forall y \in X.$$

For any $y \in X$, we must have $\phi(y, y) \notin -\mathrm{int}C$. Otherwise $\phi(y, y) \in -\mathrm{int}C$, then by (iv) and Lemma 1.4,

$$\psi(y, y) \in \phi(y, y) - C \subset -\mathrm{int}C - C \subset -\mathrm{int}C,$$

which contradicts the condition (i). It follows from $\phi(y, y) \notin -\mathrm{int}C$ that $F(y) \neq \emptyset$. By (iii) and Corollary 1.6, we know $F(y)$ is closed in $X$, which as well as the compactness of $X$ implies that $F(y)$ is compact for each $y \in X$.

Next, we shall prove that $F$ is a KKM mapping, i.e., for any finite subset $\{y_1, y_2, \cdots, y_n\}$ of $X$, we have $\mathrm{co}\{y_1, y_2, \cdots, y_n\} \subset \cup_{i=1}^n F(y_i)$. Assume that it is not true, then there exists a finite subset $\{y_1, y_2, \cdots, y_n\}$ of $X$ and $\alpha_i \geq 0, i = 1, 2, \cdots, n$, with $\sum_{i=1}^n \alpha_i = 1$, such that $x_0 := \sum_{i=1}^n \alpha_i y_i \notin \cup_{i=1}^n F(y_i)$. It follows from the convexity of $X$ that $x_0 \in X$. By the definition of $F$, we have $\phi(x_0, y_i) \in -\mathrm{int}C$ for each $i = 1, 2, \cdots, n$. Furthermore, by (iv) and Lemma 1.4, we have

$$\psi(x_0, y_i) \in \phi(x_0, y_i) - C \subset -\mathrm{int}C - C \subset -\mathrm{int}C.$$

Let

$$G := \{y \in X : \psi(x_0, y) \in -\mathrm{int}C\},$$

then it follows that $y_1, y_2, \cdots, y_n \in G$. Meanwhile the condition (ii) and Lemma 1.7 show that the set $G$ is convex, which implies $x_0 = \sum_{i=1}^{n} \alpha_i y_i \in G$, that is, $\psi(x_0, x_0) \in -\text{int}C$. This is in contradiction with the condition i), then $F$ must be a KKM mapping.

By KKMF lemma, we have $\cap_{y \in X} F(y) \neq \emptyset$. Take $x^* \in \cap_{y \in X} F(y)$, then $x^* \in F(y)$ for all $y \in X$. Thus, we have $\phi(x^*, y) \notin -\text{int}C$ for all $y \in X$.

2) Note that

$$\begin{aligned} S &= \{x \in X : \phi(x, y) \notin -\text{int}C, \ \forall y \in X\} \\ &= \cap_{y \in X} \{x \in X : \phi(x, y) \notin -\text{int}C\} \\ &= \cap_{y \in X} F(y). \end{aligned}$$

Since $F(y)$ is compact for each $y \in X$, $S$ is also compact. The proof is thus complete. ∎

In Theorem 2.1, let $\psi(x, y) = \phi(x, y)$ for all $x, y \in X$, then we get the following theorem.

**Theorem 2.2** *Let $X$ be a nonempty, convex, and compact subset of a Hausdorff topological vector space $E$. If $\phi : X \times X \to H$ satisfies the following conditions:*

(i) *for each $y \in X$, $\phi(y, y) \notin -\text{int}C$;*

(ii) *for each $y \in X$, $x \to \phi(x, y)$ is $C$-upper semi-continuous; and*

(iii) *for each $x \in X$, $y \to \phi(x, y)$ is $C$-quasiconvex.*

*Then*

1) *there exists $x^* \in X$ such that $\phi(x^*, y) \notin -\text{int}C$ for all $y \in X$; and*

2) *the solution set $S = \{x \in X : \phi(x, y) \notin -\text{int}C, \ \forall y \in X\}$ is compact.*

**Remark 2.3** Theorem 2.2 1) is a generalization of Theorem 1.1 in [15] since we suppose $y \to \phi(x, y)$ is $C$-quasiconvex instead of $C$-convex.

In the above two theorems, $X$ is compact. Next we will remove the compactness of $X$, then some coercive conditions will be considered.

**Theorem 2.4** *Let $X$ be a nonempty and convex subset of a Hausdorff topological vector space $E$. If $\phi : X \times X \to H$ satisfies the following conditions:*

(i) *for each $y \in X$, $\phi(y, y) \notin -\text{int}C$;*

(ii) *for each $y \in X$, $x \to \phi(x, y)$ is $C$-upper semi-continuous;*

(iii) *for each $x \in X$, $y \to \phi(x, y)$ is $C$-quasiconvex; and*

(iv) *there is a sequence of nonempty, convex, and compact subsets $\{X_n\}_{n=1}^{\infty}$ of $E$ with $X_1 \subset X_2 \subset X_3 \subset \cdots, X = \cup_{n=1}^{\infty} X_n$, and for each sequence $\{x_n\}_{n=1}^{\infty}$ with $x_n \in X_n$ and $\forall n, \exists x_m \notin X_n$, there exist a positive integer $n_0$ and a point $y_{n_0} \in X_{n_0}$ such that $\phi(x_{n_0}, y_{n_0}) \in -\text{int}C$.*

*Then*

1) *there exists $x^* \in X$ such that $\phi(x^*, y) \notin -\text{int}C$ for all $y \in X$; and*

2) *the solution set $S = \{x \in X : \phi(x, y) \notin -\text{int}C, \forall y \in X\}$ is compact.*

*Proof* 1) For each $n = 1, 2, \cdots$, since $X_n$ is nonempty, convex, and compact, by Theorem 2.2, there exists $x_n \in X_n$ such that $\phi(x_n, y) \notin -\text{int}C$ for all $y \in X_n$.

For the sequence $\{x_n\}_{n=1}^{\infty}$ in $X$, we show that there exists a positive integer $N_1$ such that $\{x_n\}_{n=1}^{\infty} \subset X_{N_1}$. Otherwise, for each $n$, there exists $x_m \notin X_n$. By (iv), there exist a positive integer $n_0$ and $y_{n_0} \in X_{n_0}$ such that $\phi(x_{n_0}, y_{n_0}) \in -\text{int}C$, which contradicts that $\phi(x_{n_0}, y) \notin -\text{int}C$ for all $y \in X_{n_0}$. Hence there exists a positive integer $N_1$ such that $\{x_n\}_{n=1}^{\infty} \subset X_{N_1}$. Without loss of generality, we can suppose $x_n \to x^* \in X_{N_1} \subset X$ for the sake of the compactness of $X_{N_1}$.

For any $y \in X$, since $X = \cup_{n=1}^{\infty} X_n$, there exists a positive integer $N_2$ such that $y \in X_{N_2}$. Not that

$$x_n \in \{x \in X_{N_2} : \phi(x, y) \notin -\text{int}C\} \subset \{x \in X : \phi(x, y) \notin -\text{int}C\},$$

whenever $n > N_2$. By (ii) and Corollary 1.6 1), $\{x \in X : \phi(x, y) \notin -\text{int}C\}$ is closed in $X$. Since $x_n \to x^*$, it holds $x^* \in \{x \in X : \phi(x, y) \notin -\text{int}C\}$, that is, $\phi(x^*, y) \notin -\text{int}C$ for all $y \in X$.

2) For each $y \in X$, let $F(y) := \{x \in X : \phi(x, y) \notin -\text{int}C\}$. Through the same proof as that of Theorem 2.3 2), we have that $S = \cap_{y \in X} F(y)$ and $F(y)$ is nonempty and closed in $X$ for each $y \in X$, and so $S$ is closed in $X$. Moreover, we can assert that there exists a positive integer $N$ such that $S \subset X_N$. Otherwise, there exists a sequence $\{x_n\}_{n=1}^{\infty}$ satisfying $x_n \in X_n \cap S$ and $\forall n, \exists x_m \notin X_n$. By (iv), there exist a positive integer $n_0$ and $y_{n_0} \in X_{n_0} \subset X$ such that $\phi(x_{n_0}, y_{n_0}) \in -\text{int}C$. But $x_{n_0} \in S$ implies $\phi(x_{n_0}, y) \notin -\text{int}C$ for all $y \in X$, which is a contradiction. Therefore, there exists a positive integer $N$ such that $S \subset X_N$. By the compactness of $X_N$, $S$ is compact. The proof is thus complete. ∎

It is well-known that the reflexive Banach space possesses some good properties. Let's recall them briefly. Suppose $\{x_n\}_{n=1}^{\infty}$ is a sequence of a reflexive Banach space $RB$. If $\{x_n\}_{n=1}^{\infty}$ converges to $x$ (i.e., $\|x_n - x\| \to 0$), then it must be weakly convergent to $x$ (i.e., $f(x_n) \to f(x)$ for any linear continuous functional $f$ over $RB$), but the converse is not true. $X \subset RB$ is called weakly closed, if for any sequence $\{x_n\}_{n=1}^{\infty} \subset X$, that $\{x_n\}_{n=1}^{\infty}$ weakly converges to $x$ implies $x \in X$. As is known to us, if $X$ is a weakly closed subset of $RB$, then it must be closed in $RB$, but the converse is still not true. However, when $X$ is a convex subset of $RB$, the closeness of $X$ is equivalent to its weak closeness. Moreover, if $X \subset E$ is bounded, closed, and convex, then it must be weakly compact in $E$, i.e., for any sequence $\{x_n\}_{n=1}^{\infty} \subset X$, there exists a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ such that $\{x_{n_k}\}_{k=1}^{\infty}$ weakly converges to a point $x \in X$. Especially, $R^n$ is a typical example of the reflexive Banach space, and the Hilbert space is another typical example. The following three theorems are discussed in the reflexive Banach space.

**Theorem 2.5** *Let $X$ be a nonempty and convex subset of a reflexive Banach space $RB$. If $\phi : X \times X \to H$ satisfies the following conditions:*

(i) *for each $y \in X$, $\phi(y, y) \notin -\text{int}C$;*

(ii) *for each $y \in X$, $x \to \phi(x, y)$ is weakly $C$-upper semi-continuous, here we use the weak topology in $RB$;*

(iii) *for each $x \in X$, $y \to \phi(x, y)$ is $C$-quasiconvex; and*

(iv) *there is a sequence of nonempty, bounded, closed, and convex subsets $\{X_n\}_{n=1}^{\infty}$ of $RB$ with $X_1 \subset X_2 \subset X_3 \subset \cdots, X = \cup_{n=1}^{\infty} X_n$, and for each sequence $\{x_n\}_{n=1}^{\infty}$ with $x_n \in X_n$ and $\|x_n\| \to \infty$, there exist a positive integer $n_0$ and a point $y_{n_0} \in X_{n_0}$ such that $\phi(x_{n_0}, y_{n_0}) \in -\text{int}C$.*

*Then*

1) *there exists* $x^* \in X$ *such that* $\phi(x^*, y) \notin -\mathrm{int}C$ *for all* $y \in X$*; and*

2) *the solution set* $S = \{x \in X : \phi(x, y) \notin -\mathrm{int}C, \forall y \in X\}$ *is weakly compact.*

*Proof* For each $n = 1, 2, \cdots$, since $X_n$ is a nonempty bounded closed convex subset of the reflexive Banach space $RB$, it must be weakly compact.

For each sequence $\{x_n\}_{n=1}^{\infty}$ with $x_n \in X_n$ and $\|x_n\| \to \infty$, we claim that for each $n = 1, 2, \cdots$, there must exist $x_m \notin X_n$. Otherwise there exists a positive integer $N$ such that $\{x_n\}_{n=1}^{\infty} \subset X_N$. But $\|x_n\| \to \infty$ contradicts that $X_N$ is bounded. So (iv) implies the condition (iv) of Theorem 2.4. By Theorem 2.4, this theorem holds. ∎

**Theorem 2.6** *Let* $X$ *be a nonempty, closed, and convex subset of a reflexive Banach space* $RB$*. If* $\phi : X \times X \to H$ *satisfies the following conditions:*

(i) *for each* $y \in X$, $\phi(y, y) \notin -\mathrm{int}C$*;*

(ii) *for each* $y \in X$, $x \to \phi(x, y)$ *is weakly* $C$*-upper semi-continuous, here we use the weak topology in* $RB$*;*

(iii) *for each* $x \in X, y \to \phi(x, y)$ *is* $C$*-quasiconvex; and*

(iv) *given* $\widehat{x} \in E$*, for each sequence* $\{x_n\}_{n=1}^{\infty} \subset X$ *with* $\|x_n\| \to \infty$*, there exist a positive integer* $n_0$ *and a point* $y_{n_0} \in X$ *such that* $\|y_{n_0} - \widehat{x}\| \leq \|x_{n_0} - \widehat{x}\|$ *and* $\phi(x_{n_0}, y_{n_0}) \in -\mathrm{int}C$*. Then*

1) *there exists* $x^* \in X$ *such that* $\phi(x^*, y) \notin -\mathrm{int}C$ *for all* $y \in X$*; and*

2) *the solution set* $S = \{x \in X : \phi(x, y) \notin -\mathrm{int}C, \forall y \in X\}$ *is weakly compact.*

*Proof* 1) For each $n = 1, 2, \cdots$, set

$$X_n = \{x \in X : \|x - \widehat{x}\| \leq n\} = X \cap \{x \in E : \|x - \widehat{x}\| \leq n\}.$$

We may suppose $X_n$ is nonempty for each $n$. Since $X$ is convex and closed, then $X_n$ is bounded, closed, and convex in the reflexive Banach space $RB$, so it is weakly compact and convex. It is obvious that $\{X_n\}_{n=1}^{\infty}$ satisfying $X_1 \subset X_2 \subset X_3 \subset \cdots$, and $X = \cup_{n=1}^{\infty} X_n$.

Let $\{x_n\}_{n=1}^{\infty}$ be any sequence in $X$ with $x_n \in X_n$ and $\|x_n\| \to \infty$. Then by (iv), there exist a positive integer $n_0$ and a point $y_{n_0} \in X$ such that

$$\|y_{n_0} - \widehat{x}\| \leq \|x_{n_0} - \widehat{x}\| \text{ and } \phi(x_{n_0}, y_{n_0}) \in -\mathrm{int}C.$$

From $\|y_{n_0} - \widehat{x}\| \leq \|x_{n_0} - \widehat{x}\| \leq n_0$, we know $y_{n_0} \in X_{n_0}$. Hence the condition (iv) of Theorem 2.5 holds. By Theorem 2.5, there exists $x^* \in X$ such that $\phi(x^*, y) \notin -\mathrm{int}C$ for all $y \in X$.

2) Through the same proof as that of Theorem 2.1, we can show that $S$ is closed in $X$ with respect to the weak topology, i.e., $S$ is weakly closed in $X$. Assume that $S$ is unbounded, then there exists a sequence $\{x_n\}_{n=1}^{\infty}$ with $x_n \in S \subset X$ and $\|x_n\| \to \infty$. By (iv), there exists a positive integer $n_0$ and $y_{n_0} \in X$ such that

$$\|y_{n_0} - \widehat{x}\| \leq \|x_{n_0} - \widehat{x}\| \leq n_0,$$

i.e.,

$$y_{n_0} \in X_{n_0} \text{ and } \phi(x_{n_0}, y_{n_0}) \in -\mathrm{int}C,$$

which contradicts that

$$x_{n_0} \in S \text{ and } \phi(x_{n_0}, y) \notin -\text{int}C \text{ for all } y \in X.$$

Therefore, $S$ must be bounded, that is, there exists a positive integer $N$ such that $S \subset X_N$. By the weak compactness of $X_N$, we have that $S$ is weakly compact. ▮

If there is no need to ensure the compactness of the solution set, Theorem 2.6 can be adjusted to the following theorem.

**Theorem 2.7** *Let $X$ be a nonempty, closed, and convex subset of a reflexive Banach space $RB$. If $\phi : X \times X \to H$ satisfies the following conditions:*

(i) *for each $y \in X, \phi(y, y) \notin -\text{int}C$;*

(ii) *for each $y \in X, x \to \phi(x, y)$ is weakly $C$-upper semicontinuous, here we use the weak topology in $RB$;*

(iii) *for each $x \in X, y \to \phi(x, y)$ is $C$-convex; and*

(iv) *given $\widehat{x} \in RB$, for each sequence $\{x_n\}_{n=1}^{\infty} \subset X$ with $\|x_n\| \to \infty$, there exist a positive integer $n_0$ and a point $y_{n_0} \in X$ such that $\|y_{n_0} - \widehat{x}\| < \|x_{n_0} - \widehat{x}\|$ and $\phi(x_{n_0}, y_{n_0}) \in -C$.*
*Then there exists $x^* \in X$ such that $\phi(x^*, y) \notin -\text{int}C$ for all $y \in X$.*

*Proof* By Remark 1.9, (iii) implies (iii) of Theorem 2.6. Assume that the VKFI $\phi$ has no solution. Under this assumption, we will prove that (iv) implies (iv) of Theorem 2.6. Then by Theorem 2.6, there exists $x^* \in X$ such that $\phi(x^*, y) \notin -\text{int}C$ for all $y \in X$, which is a contradiction with the assumption.

Indeed, suppose that (iv) holds, i.e., for each sequence $\{x_n\}_{n=1}^{\infty}$ with $\|x_n\| \to \infty$, there exist a positive integer $n_0$ and a point $y_{n_0} \in X$ such that

$$\|y_{n_0} - \widehat{x}\| < \|x_{n_0} - \widehat{x}\| \text{ and } \phi(x_{n_0}, y_{n_0}) \in -C.$$

From the assumption, we know that $x_{n_0}$ is not a solution of VKFI $\phi$, so there exists $u_0 \in X$ such that $\phi(x_{n_0}, u_0) \in -\text{int}C$. Let $\lambda_0 > 0$ be so small that

$$z_{n_0} := \lambda_0 u_0 + (1 - \lambda_0) y_{n_0} \in X$$

satisfies $\|z_{n_0} - \widehat{x}\| \leq \|x_{n_0} - \widehat{x}\|$. Since $y \to \phi(x_{n_0}, y)$ is $C$-convex, we have

$$\begin{aligned}\phi(x_{n_0}, z_{n_0}) &= \phi(x_{n_0}, \lambda_0 u_0 + (1 - \lambda_0) y_{n_0}) \in \lambda_0 \phi(x_{n_0}, u_0) + (1 - \lambda_0) \phi(x_{n_0}, y_{n_0}) - C \\ &\subset -\lambda_0(\text{int}C) - (1 - \lambda_0)C - C \subset -\text{int}C - C \subset -\text{int}C,\end{aligned}$$

which shows that (iv) of Theorem 2.6 holds. Thus the proof is completed. ▮

**Remark 2.8** We must emphasize that the conditions (iii) and (iv) of Theorem 2.6 are different from that of Theorem 2.7, and hence Theorem 2.7 does not claim that the solution set is weakly compact.

For the conditions (iv) of Theorems 2.6 and 2.7, we have the following two lemmas.

**Lemma 2.9** *Let $X$ be a nonempty subset of a normed space $E$ and $\widehat{x} \in E$. Then the following two statements are equivalent:*

(iv) *For each sequence* $\{x_n\}_{n=1}^{\infty} \subset X$ *with* $\|x_n\| \to \infty$, *there exist a positive integer* $n_0$ *and a point* $y_{n_0} \in X$ *such that* $\|y_{n_0} - \widehat{x}\| \leq \|x_{n_0} - \widehat{x}\|$ *and* $\phi(x_{n_0}, y_{n_0}) \in -\mathrm{int}C$.

(iv′) *There exists a constant* $\rho > 0$ *such that for any* $x \in X$ *with* $\|x - \widehat{x}\| > \rho$, *there exists* $y \in X$ *satisfying* $\|y - \widehat{x}\| \leq \|x - \widehat{x}\|$ *and* $\phi(x, y) \in -\mathrm{int}C$.

*Proof* (iv)⇒ (iv′) Assume (iv′) doesn't hold, then there exists a sequence $\{x_n\}_{n=1}^{\infty} \subset X$ with $\|x_n\| \to \infty$ such that for each $n$ and any $y \in X$, if $\|y - \widehat{x}\| \leq \|x_n - \widehat{x}\|$ then $\phi(x_n, y) \notin -\mathrm{int}C$. But it follows from (iv) that there exist a positive integer $n_0$ and a point $y_{n_0} \in X$ such that $\|y_{n_0} - \widehat{x}\| \leq \|x_{n_0} - \widehat{x}\|$ and $\phi(x_{n_0}, y_{n_0}) \in -\mathrm{int}C$, which is a contradiction. Hence (iv′) must hold.

(iv′)⇒ (iv) Suppose the sequence $\{x_n\}_{n=1}^{\infty} \subset X$ satisfies $\|x_n\| \to \infty$, then for any $\rho > 0$, there exists a positive integer $n_0$ such that $\|x_{n_0} - \widehat{x}\| > \rho$. Since (iv′) holds, there exists a point $y_{n_0} \in X$ such that $\|y_{n_0} - \widehat{x}\| \leq \|x_{n_0} - \widehat{x}\|$ and $\phi(x_{n_0}, y_{n_0}) \in -\mathrm{int}C$, which shows that (iv) does hold. ∎

**Lemma 2.10** *Let* $X$ *be a nonempty subset of a normed space* $E$ *and* $\widehat{x} \in E$. *Then the following two statements are equivalent:*

(iv) *For each sequence* $\{x_n\}_{n=1}^{\infty} \subset X$ *with* $\|x_n\| \to \infty$, *there exist a positive integer* $n_0$ *and a point* $y_{n_0} \in X$ *such that* $\|y_{n_0} - \widehat{x}\| < \|x_{n_0} - \widehat{x}\|$ *and* $\phi(x_{n_0}, y_{n_0}) \in -C$.

(iv′) *There exists a constant* $\rho > 0$ *such that for any* $x \in X$ *with* $\|x - \widehat{x}\| > \rho$, *there exists* $y \in X$ *satisfying* $\|y - \widehat{x}\| < \|x - \widehat{x}\|$ *and* $\phi(x, y) \in -C$.

*Proof* The proof is similar with that of Lemma 2.9. ∎

**Remark 2.11** If we replace (iv) by (iv′) in Theorems 2.6 and 2.7, respectively, we can obtain some other theorems for VKFIs, which can be simply written out and so are omitted.

## 3 Applications to Vector Variational Inequalities

We emphasize again that in the whole paper $H$ denotes a Hausdorff topological vector space and $C$ denotes a nonempty closed, convex, and pointed cone in $H$ with $\mathrm{int}C \neq \emptyset$. Denote by $L(E, H)$ the space of all linear continuous operators from a Hausdorff topological vector space $E$ to $H$. For $l \in L(E, H)$, the value of linear operators $l$ at $x$ is denoted by $\langle l, x\rangle$. A vector variational inequality (for short, VVI)[6] is a problem of finding $x^* \in X$ such that

$$\langle T(x^*), y - x^*\rangle \notin -\mathrm{int}C, \ \forall y \in X,$$

where $T : X \to L(E, H)$ and $X$ is a nonempty subset of $E$.

**Theorem 3.1** *Let* $X$ *be a nonempty, convex, and compact subset of a Hausdorff topological vector space* $E$. *If* $T : X \to L(E, H)$ *is continuous, then*

1) *there exists* $x^* \in X$ *such that* $\langle T(x^*), y - x^*\rangle \notin -\mathrm{int}C$ *for all* $y \in X$*; and*

2) *the solution set* $V = \{x \in X : \langle T(x), y - x\rangle \notin -\mathrm{int}C, \ \forall y \in X\}$ *is compact.*

*Proof* Define $\phi : X \times X \to H$ as following

$$\phi(x, y) = \langle T(x), y - x\rangle, \quad \forall x, y \in X.$$

Then for each $y \in X$, $\phi(y,y) = 0 \notin -\mathrm{int}C$; for each $y \in X$, $x \to \phi(x,y)$ is continuous, so it must be $C$-upper semi-continuous; and it is obvious that for each $x \in X$, $y \to \phi(x,y)$ is affine, i.e.,

$$\forall y_1, y_2 \in X,\ \forall \lambda \in R,\ \phi(x, \lambda y_1 + (1-\lambda)y_2) = \lambda\phi(x,y_1) + (1-\lambda)\phi(x,y_2),$$

hence it is $C$-quasiconvex. Note that for each $y \in X$, $\langle T(x^*), y - x^*\rangle \notin -\mathrm{int}C$ if and only if $\phi(x^*, y) \notin -\mathrm{int}C$. By applying Theorem 2.2, the proof is easily finished. ∎

The following Theorems 3.2–3.5 can be derived from Theorems 2.4–2.7 respectively by proofs analogous to that of Theorem 3.1 and their proofs are thus omitted.

**Theorem 3.2** *Let $X$ be a nonempty and convex subset of a Hausdorff topological vector space $E$ and $T : X \to L(E,H)$ be continuous. Suppose there is a sequence of nonempty, convex, and compact subsets $\{X_n\}_{n=1}^{\infty}$ of $E$ with $X_1 \subset X_2 \subset X_3 \subset \cdots, X = \cup_{n=1}^{\infty} X_n$ and suppose that for each sequence $\{x_n\}_{n=1}^{\infty}$ with $x_n \in X_n$ and $\forall n, \exists x_m \notin X_n$, there exist a positive integer $n_0$ and a point $y_{n_0} \in X_{n_0}$ such that $\langle T(x_{n_0}), y_{n_0} - x_{n_0}\rangle \in -\mathrm{int}C$. Then*

1) *there exists $x^* \in X$ such that $\langle T(x^*), y - x^*\rangle \notin -\mathrm{int}C$ for all $y \in X$; and*
2) *the solution set $V = \{x \in X : \langle T(x), y - x\rangle \notin -\mathrm{int}C, \forall y \in X\}$ is compact.*

**Theorem 3.3** *Let $X$ be a nonempty and convex subset of a reflexive Banach space $RB$, and $T : X \to L(RB,H)$ be continuous. Suppose there is a sequence of nonempty, bounded, closed, and convex subsets $\{X_n\}_{n=1}^{\infty}$ of $RB$ with $X_1 \subset X_2 \subset X_3 \subset \cdots, X = \cup_{n=1}^{\infty} X_n$ and suppose that for each sequence $\{x_n\}_{n=1}^{\infty}$ with $x_n \in X_n$ and $\|x_n\| \to \infty$, there exist a positive integer $n_0$ and a point $y_{n_0} \in X_{n_0}$ such that $\langle T(x_{n_0}), y_{n_0} - x_{n_0}\rangle \in -\mathrm{int}C$. Then*

1) *there exists $x^* \in X$ such that $\langle T(x^*), y - x^*\rangle \notin -\mathrm{int}C$ for all $y \in X$; and*
2) *the solution set $V = \{x \in X : \langle T(x), y - x\rangle \notin -\mathrm{int}C,\ \forall y \in X\}$ is weakly compact.*

**Theorem 3.4** *Let $X$ be a nonempty, closed, and convex subset of a reflexive Banach space $RB$, $T : X \to L(RB,H)$ be continuous, and $\widehat{x} \in E$ be a given vector. Suppose that for each sequence $\{x_n\}_{n=1}^{\infty} \subset X$ with $\|x_n\| \to \infty$, there exist a positive integer $n_0$ and a point $y_{n_0} \in X$ such that $\|y_{n_0} - \widehat{x}\| \le \|x_{n_0} - \widehat{x}\|$ and $\langle T(x_{n_0}), y_{n_0} - x_{n_0}\rangle \in -\mathrm{int}C$. Then*

1) *there exists $x^* \in X$ such that $\langle T(x^*), y - x^*\rangle \notin -\mathrm{int}C$ for all $y \in X$, and*
2) *the solution set $V = \{x \in X : \langle T(x), y - x\rangle \notin -\mathrm{int}C,\ \forall y \in X\}$ is weakly compact.*

**Theorem 3.5** *Let $X$ be a nonempty, closed and convex subset of a reflexive Banach space $RB$, $T : X \to L(RB,H)$ be continuous, and $\widehat{x} \in E$ be a given vector. Suppose that for each sequence $\{x_n\}_{n=1}^{\infty} \subset X$ with $\|x_n\| \to \infty$, there exists a positive integer $n_0$ and a point $y_{n_0} \in X$ such that $\|y_{n_0} - \widehat{x}\| < \|x_{n_0} - \widehat{x}\|$ and $\langle T(x_{n_0}), y_{n_0} - x_{n_0}\rangle \in -C$. Then there exists $x^* \in X$ such that $\langle T(x^*), y - x^*\rangle \notin -\mathrm{int}C$ for all $y \in X$.*

**Remark 3.6** Let $E$ be a Hilbert space, $X \subset E$, $H = R$, then $L(E,R) = E^*$ denotes the space of all linear continuous functionals on $E$. The mapping $T : X \to E^*$ means that for each $x \in X$, $T(x)$ is a linear continuous functionals on $E$. As is known to us, each linear continuous functionals on the Hilbert space $E$ is decided by a unique element of $E$ so that $T : X \to E^*$ can be considered as $T : X \to E$. Moreover, if let $C = [0, +\infty)$, then $-\mathrm{int}C = (-\infty, 0)$. Thus the VVI becomes a variational inequality of finding $x^* \in X$ such that

$$\langle T(x^*), y - x^* \rangle \geq 0, \ \forall y \in X,$$

where $T : X \to E$ and $\langle \cdot, \cdot \rangle$ is the inner product on $E$.

**Remark 3.7** Theorems 3.4–3.5 are partly inspired by the results in [18, 19], meanwhile they are vectorial version and infinite dimensional generalizations of Theorem 3.1 in [18] and Theorem 2.4.27 in [19]. It's important that Theorem 3.1 in [18] and Theorem 2.4.27 in [19] are based on the profound analysis of the topological degree and exceptional family, while we only apply the VKFIs which are based on the well-known KKMF lemma in nonlinear analysis to present the results directly and easily.

**Remark 3.8** Our results in Section 2 can also be applied conveniently to study the noncooperative multi-objective games.

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