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Existence and stability analysis of optimal control

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SUMMARY

By constructing a complete metric space and a compact set of admissible control functions, this paper investigates the existence and stability of solutions of optimal control problems with respect to the right-hand side functions. On the basis of set-valued mapping theory, by introducing the notion of essential solutions for optimal control problems, some sufficient and necessary criteria guaranteeing the existence and stability of solutions are established. New derived criteria show that the optimal control problems whose solutions are all essential form a dense residual set, and so every optimal control problem can be closely approximated arbitrarily by an essential optimal control problem. The example shows that not all optimal control problems are stable. However, our main result shows that, in the sense of Baire category, most of the optimal control problems are stable. Copyright © 2013 John Wiley & Sons, Ltd.

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1. INTRODUCTION

Beginning in the late 1950s and continuing today, the issues concerning dynamic optimization have received much attention because of its potential applications in all kinds of branches of science, such as optimal control [1–4] and differential game theory [5–7]. The impact of optimal control is witnessed by the magnitude of the work and the number of results that have been obtained, spanning theoretical aspects as well as applications. The aim of optimal control problems is to find a controller for a given system such that a certain optimality criterion (performance index) is achieved. In the standard model of control theory, the state of a system is described by a function $x : [t_0, T] \to \mathbb{R}^m$. This state evolves in time, and we have the ability to influence its evolution through a vector-valued control $u(t) \in \mathbb{R}^n$. The evolution of the system is determined by a system of ordinary differential equations

$$\begin{cases} \dot{x}(t) = f(t, x(t), u(t)), & t \in [t_0, T], \\ x(t_0) = x^0, \end{cases} \tag{1}$$

where the right-hand function f is also a vector-valued function. The Bolza optimal control problem is to choose a control function (solution) $u^*(t)$ with $t \in [t_0, T]$ so as to minimize the cost

$$h(x(T)) + \int_{t_0}^{T} g(t, x(t), u(t))\mathrm{d}t, \tag{2}$$

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where h is a terminal payoff and g accounts for a running cost. That is,

$$h(x^*(T)) + \int_{t_0}^{T} g(t, x^*(t), u^*(t))\mathrm{d}t = \min_{u \in U} \left\{ h(x(T)) + \int_{t_0}^{T} g(t, x(t), u(t))\mathrm{d}t \right\}, \tag{3}$$

where U is a set of admissible control functions and $x(t)$ is determined uniquely by (1). We call such an optimal control function $u^*(t)$ a solution for the optimal control problem. Obviously, the solution $u^*(t)$ relies on the right-hand function f, which decides the state equation (1). In previous works, the right-hand sidefunction f is always a given function satisfying some assumptions. In contrast, the focus in this paper is on another, natural question: if f changes slightly, what are the impacts on the solution $u^*(t)$? Does it still exist? Will the set of solutions have a big change? These problems all concerned with the stability of solutions of optimal control problems with respect to the right-hand side functions.

In fact, the research on the stability of solutions of nonlinear problems has led to many achievements. For instance, Wu and Jiang [8] introduced the notion of essential equilibria in finite non-cooperative games and proved that any finite non-cooperative game can be approximated arbitrarily by a game whose equilibria are all essential. Yu [9] and Carbonell-Nicolau [10] extended the notion and the results to general n-person non-cooperative games. Recently, the concept of essentiality has been widely used in studying the stability of solutions in various nonlinear problems including optimization problems, fixed point problems, Ky Fans point problems, Nash equilibrium problems, and KKM point problems (see, for instance, [11–15]).

In this article, we study the robustness of solutions against the perturbation of the right-hand side functions of optimal control problems. By constructing a compact set of admissible control functions and a complete metric space, we consider the existence and stability of the solutions of the typical optimal control problems. On the basis of set-valued mapping theory, some sufficient and necessary criteria guaranteeing the existence and stability of solutions are established. New derived criteria show that the optimal control problems whose solutions are all essential form a dense residual set, and so every optimal control problem can be closely approximated arbitrarily by an essential optimal control problem. One perhaps wonders whether all the optimal control problems are essential. An example is provided to show that it is not true. But our results show that, in the sense of Baire category, most of the optimal control problems are essential.

2. MODELS AND PRELIMINARIES

To investigate the stability of solutions for optimal control problems, we need some topological structures on the spaces of functions and on the spaces of solutions.

Let $t_0, T \in \mathbb{R}$ with $0 \leqslant t_0 < T$, and

$$C_n[t_0, T] \triangleq \left\{ u = (u_1, u_2, \cdots, u_n) : [t_0, T] \to \mathbb{R}^n : \begin{array}{c} \forall i = 1, 2, \cdots, n, \\ u_i \text{ is continuous.} \end{array} \right\}. \tag{4}$$

For any $u \in C_n[t_0, T]$, define

$$\|u\| \triangleq \max_{t_0 \leqslant t \leqslant T} \|u(t)\|.$$

Let U be a nonempty and closed subset of $C_n[t_0, T]$ such that

(**A**1) U is uniformly bounded with the constant $K > 0$, namely, $\|u\| \leqslant K$ for all $u \in U$; and
(**A**2) U is equicontinuous, that is, for any $\varepsilon > 0$, there exists $\delta > 0$ such that for any $t_1, t_2 \in [t_0, T]$ with $|t_1 - t_2| < \delta$ and any $u \in U$, it holds that $\|u(t_1) - u(t_2)\| < \varepsilon$.

From (A1), (A2) and Ascoli-Arzela theorem, U is a compact subset of $C_n[t_0, T]$.

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Let $M > 0, L > 0, G > 0$, $B \triangleq \{u \in \mathbb{R}^n : \|u\| \leqslant K\}$, $D \triangleq [t_0, T] \times \mathbb{R}^m \times B$, and

$$Y \triangleq \left\{ f = (f_1, f_2, \cdots, f_m) : D \to \mathbb{R}^m : \begin{array}{l} \forall i = 1, 2, \cdots, m,\ f_i \text{ is continuous,} \\ \sup\limits_{(t,x,u)\in D} \|f(t,x,u)\| \leqslant M, \text{ and} \\ \|f(t,x_1,u) - f(t,x_2,u)\| \leqslant L\|x_1 - x_2\| \\ \text{for any } (t,x_1,u), (t,x_2,u) \in D. \end{array} \right\} \tag{5}$$

For each $f^1 = \left(f_1^1, f_2^1, \cdots, f_m^1\right), f^2 = \left(f_1^2, f_2^2, \cdots, f_m^2\right) \in Y$, define

$$\rho\left(f^1, f^2\right) \triangleq \sup_{(t,x,u)\in D} \|f^1(t,x,u) - f^2(t,x,u)\|. \tag{6}$$

Then, one can easily prove that (Y, ρ) is a complete metric space.

Given the initial value $x^0 \in \mathbb{R}^n$, for each $f \in Y$ and each $u \in U$, according to the existence and uniqueness theorem and the extension theorem of differential equations, the system of differential equations (1) has a unique solution $x : [t_0, T] \to \mathbb{R}^n$. Obviously, x depends not only on f but also on u. This implies that x can be denoted by $x(\cdot) = A_f(u)(\cdot)$. In what follows, we first cite an important lemma and then give a basic proposition.

Lemma 2.1 (Gronwall inequality [16]*)*
Let $w(t) \geqslant 0$ be continuous on $[a, +\infty)$ and $c \geqslant 0$ be a constant. If

$$w(t) \leqslant c + \int_a^t w(s)\mathrm{d}s, \forall t \in [a, +\infty),$$

then

$$w(t) \leqslant ce^{(t-a)}, \forall t \in [a, +\infty).$$

Proposition 2.1
$\forall f^k \in Y, f^k \to f \in Y, \forall u^k \in U, u^k \to u$, then $A_{f^k}(u^k) \to A_f(u)$.

Proof
Denote by $x^k = A_{f^k}(u^k), x = A_f(u)$. By (1), for $t \in [t_0, T]$,

$$\begin{cases} x^k(t) = x^0 + \displaystyle\int_{t_0}^t f^k\left(s, x^k(s), u^k(s)\right)\mathrm{d}s, \\ x(t) = x^0 + \displaystyle\int_{t_0}^t f(s, x(s), u(s))\mathrm{d}s. \end{cases} \tag{7}$$

Because $f^k \to f$, for any $\varepsilon > 0$, there exists $N_1 > 0$ such that $\rho\left(f^k, f\right) < \frac{\varepsilon}{2}$ whenever $k \geqslant N_1$. Notice that $x(s)$ is continuous on $[t_0, T]$, there is $a > 0$ such that $\max\limits_{t_0 \leqslant s \leqslant T} \|x(s)\| \leqslant a$. Because f is uniformly continuous on $[t_0, T] \times \{x \in \mathbb{R}^m : \|x\| \leqslant a\} \times \{u \in \mathbb{R}^n : \|u\| \leqslant K\}$, and $u^k \to u$, there exists $N_2 > 0$ such that

$$\|f\left(s, x(s), u^k(s)\right) - f(s, x(s), u(s))\| < \frac{\varepsilon}{2}$$

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for all $s \in [t_0, T]$ whenever $k \geqslant N_2$. Set $N = \max\{N_1, N_2\}$, from (7), it yields for any $k \geqslant N$ that

$$\begin{aligned}
\|x^k(t) - x(t)\| &\leqslant \int_{t_0}^{t} \|f^k\left(s, x^k(s), u^k(s)\right) - f(s, x(s), u(s))\|\mathrm{d}s \\
&\leqslant \int_{t_0}^{t} \|f^k\left(s, x^k(s), u^k(s)\right) - f\left(s, x^k(s), u^k(s)\right)\|\mathrm{d}s \\
&\quad + \int_{t_0}^{t} \|f\left(s, x^k(s), u^k(s)\right) - f\left(s, x(s), u^k(s)\right)\|\mathrm{d}s \\
&\quad + \int_{t_0}^{t} \|f\left(s, x(s), u^k(s)\right) - f(s, x(s), u(s))\|\mathrm{d}s \\
&\leqslant \int_{t_0}^{t} \frac{\varepsilon}{2}\mathrm{d}s + \int_{t_0}^{t} L\|x^k(s) - x(s)\|\mathrm{d}s \quad + \int_{t_0}^{t} \frac{\varepsilon}{2}\mathrm{d}s \\
&\leqslant L\int_{t_0}^{t} \left(\|x^k(s) - x(s)\| + \frac{\varepsilon}{L}\right)\mathrm{d}s,\ \forall t \in [t_0, T].
\end{aligned}$$

Namely, $\|x^k(t)-x(t)\| + \frac{\varepsilon}{L} \leqslant L\int_{t_0}^{t}\left(\|x^k(s)-x(s)\| + \frac{\varepsilon}{L}\right)\mathrm{d}s + \frac{\varepsilon}{L}, \forall t \in [t_0, T]$. By Lemma 2.1, for any $k \geqslant N$, it yields

$$\|x^k(t) - x(t)\| + \frac{\varepsilon}{L} \leqslant \frac{\varepsilon}{L}e^{L(t-t_0)} \leqslant \frac{\varepsilon}{L}e^{L(T-t_0)}, \forall t \in [t_0, T],$$

which means that $\|x^k(t) - x(t)\| \leqslant \frac{\varepsilon}{L}\left(\|e^{L(T-t_0)} - 1\right), \forall t \in [t_0, T]$.

From the arbitrary of $\varepsilon > 0$, it yields $x^k \to x$, that is, $A_{f^k}(u^k) \to A_f(u)$. This completes the proof. □

From Proposition 2.1, one can easily obtain the following result.

Corollary 2.1
Let $f \in Y$, $\forall u^k \in U$, $u^k \to u \in U$. Then, $A_f(u^k) \to A_f(u)$.

3. EXISTENCE AND STABILITY ANALYSIS OF OPTIMAL CONTROL

Because, from Section 2, $x(t)$ can be denoted by $x(t) = A_f(u)(t)$, the optimal control problem (3) can be describe to find $u^* \in U$ such that

$$J_f(u^*) = \min_{u \in U} J_f(u), \tag{8}$$

where

$$J_f(u) \triangleq h(x(T)) + \int_{t_0}^{T} g(t, x(t), u(t))\mathrm{d}t, \tag{9}$$

h and g are continuous, $f \in Y$ and $x = A_f(u)$. Obviously, u^* depends on f. Denote by $S(f)$ the solution set of (8) for each $f \in Y$. Then, the correspondence $f \to S(f)$ yields a set-valued mapping $S : Y \rightrightarrows U$. We shall study the stability of $S(f)$.

Now, let us recall some definitions about set-valued mappings, see [17]. Let U be a metric space. A set-valued mapping $S : Y \rightrightarrows U$ is called (1) upper (respectively, lower) semicontinuous at $f \in Y$ iff for each open set G in U with $G \supset S(f)$ (respectively, $G \cap S(f) \neq \emptyset$), there exists $\delta > 0$ such that $G \supset S(f')$ (respectively, $G \cap S(f') \neq \emptyset$) for any $f' \in Y$ with $\rho(f', f) < \delta$; (2) continuous at $f \in Y$ iff S is both upper and lower semicontinuous at f; (3) an usco mapping iff S is upper semicontinuous and $S(f)$ is nonempty compact for each $f \in Y$; and (4) closed iff Graph(S) is closed, where Graph$(S) := \{(f, u) \in Y \times U : u \in S(f)\}$ is the graph of S. Also recall that a subset $Q \subset Y$ is called a residual set iff it contains a countable intersection of open dense subsets of Y. Then, if Y is a complete metric space, any residual subset of Y must be dense in Y.

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Lemma 3.1 ([17]*)*
If $S : Y \rightrightarrows U$ is closed and U is compact, then S is upper semicontinuous at each $f \in Y$.

The following result follows from Theorem 2 in [18] or Lemma 2.3 in [13].

Lemma 3.2
Let Y be a complete metric space, U be a metric space and $S : Y \rightrightarrows U$ be an usco mapping. Then, there exists a dense residual subset Q of Y such that S is lower semicontinuous at each $f \in Q$.

Remark 3.1
A subset $Q \subset Y$ is called dense if the closure $\bar{Q}$ of Q is equal to Y; Q is called nowhere dense if the interior int($\bar{\text{Q}}$) of $\bar{Q}$ is empty; Q is called a first category set if Q can be expressed as a union of countable many nowhere dense subsets, otherwise Q is called a second category set; Q is called residual if Q contains the intersection of countable many of dense and open subsets of Y. If Y is a complete metric space, then the residual subset of Y is dense in Y and is a second category set. We refer to [19] for more details on the Baire category theory. If there exists a dense residual subset Q of Y such that for each $f \in Y$, a certain property P depending on f holds, then we say that the property P is generic on Y. Because Q is a second category set, we may say that, in the sense of Baire category, the property P holds for most of the points in Y. The research on generic properties (including generic existence, generic uniqueness, generic stability and generic well-posedness) has attracted much attention (see, for example, [9–15]). We also note that the Baire category and the category by measurements are different (see the following example).

Example 3.1
Let $0 < \alpha < 1$. At the first step, we delete the open interval $\left(\frac{1}{2} - \frac{\alpha}{4}, \frac{1}{2} + \frac{\alpha}{4}\right)$ from [0,1] with length $\frac{\alpha}{2}$ and midpoint $\frac{1}{2}$; at the second step, we delete an open interval with length $\frac{\alpha}{8}$ from $\left[0, \frac{1}{2} - \frac{\alpha}{4}\right]$ and $\left[\frac{1}{2} + \frac{\alpha}{4}, 1\right]$, respectively; at the third step, $\cdots$; then continue to do like this. Consequently, we delete a union, say $G(\alpha)$, of countable many open intervals. Note that $G(\alpha)$ is open and dense in [0,1]. We can check that the measurement of $G(\alpha)$ is $mG(\alpha) = \frac{\alpha}{2} + \frac{\alpha}{4} + \frac{\alpha}{8} + \cdots = \alpha$. Let $Q = \bigcap_{n=1}^{\infty} G\left(\frac{1}{2^n}\right)$. Then, Q is a dense residual subset of [0,1]. For each $n = 1, 2, \cdots$, $mQ \leqslant mQ\left(\frac{1}{2^n}\right) = \frac{1}{2^n}$, so $mQ = 0$, but Q is a second category set. Let $F = [0,1] \setminus Q = \bigcup_{n=1}^{\infty} \left\{[0,1] \setminus G\left(\frac{1}{2^n}\right)\right\}$. It is easy to prove that F is a first category set but $mF = 1$. Therefore, we know that

F is a first category set $\not\Longleftrightarrow mF = 0$; Q is a second category set $\not\Longleftrightarrow mQ = 1$;
$mQ = 0 \not\Longleftrightarrow Q$ is a first category set; $mF = 1 \not\Longleftrightarrow F$ is a second category set.

From now on, we begin to investigate the properties of the solution mapping S of the optimal control problems.

Theorem 3.1
$S(f) \neq \emptyset$ for each $f \in Y$.

Proof
For any fixed $f \in Y$, let $u^k \in U$ with $u^k \to u \in U$. From Corollary 2.1, it follows that $x^k = A_f(u^k) \to x = A_f(u)$. Notice that h and g are both continuous. Then, we have

$$h\left(x^k(T)\right) + \int_{t_0}^{T} g\left(t, x^k(t), u^k(t)\right) \mathrm{d}s \to h(x(T)) + \int_{t_0}^{T} g(t, x(t), u(t)) \mathrm{d}s,$$

that is, $J_f(u^k) \to J_f(u)$. This shows that $J_f(\cdot)$ is continuous on U. Because U is a compact subset, the minimizer of (8) exists, that is, $S(f) \neq \emptyset$ for each $f \in Y$. □

Lemma 3.3
Let $\{f^k\} \subset Y$ with $f^k \to f \in Y$ and $\{u^k\} \subset U$ with $u^k \to u \in U$. Then, $J_{f^k}(u^k) \to J_f(u)$.

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Proof
By (9), $J_{f^k}(u^k) = h\left(x^k(T)\right) + \int_{t_0}^{T} g\left(t, x^k(t), u^k(t)\right) \mathrm{d}t,\ \forall k = 1, 2, \cdots$, where h and g are continuous and $x^k = A_{f^k}(u^k)$. Because $f^k \to f$ and $u^k \to u$, as well as $x^k \to x$ by Proposition 2.1, then

$$h\left(x^k(T)\right) + \int_{t_0}^{T} g\left(t, x^k(t), u^k(t)\right) \mathrm{d}t \ \to\ h(x(T)) + \int_{t_0}^{T} g(t, x(t), u(t)) \mathrm{d}t,$$

that is, $J_{f^k}(u^k) \to J_f(u)$. □

Theorem 3.2
$S : Y \rightrightarrows U$ is an usco mapping.

Proof
Because U is compact, by Lemma 3.1, it suffices to show that Graph(S) is closed, where Graph(S) $:= \{(f, u) \in Y \times U : u \in S(f)\}$. Let $\{f^k\} \subset Y$ with $f^k \to f \in Y$ and $u^k \in S(f^k)$ with $u^k \to u^* \in U$. Then, let us show $u^* \in S(f)$.

In fact, for each $k = 1, 2, \cdots$, because $u^k \in S(f^k)$, we have $J_{f^k}(u^k) \leqslant J_{f^k}(u)$ for all $u \in U$. Because $f^k \to f$ and $u^k \to u^*$, from Lemma 3.3, it yields that

$$J_{f^k}(u^k) \to J_f(u^*), \text{ and } J_{f^k}(u) \to J_f(u) \text{ for all } u \in U.$$

Thus, we have $J_f(u^*) \leqslant J_f(u)$ for all $u \in U$, namely, $u^* \in S(f)$. This completes the proof. □

In order to research the stability of the solutions of optimal control problems, we introduce the following definition.

Definition 3.1
Let $f \in Y$. (1) $u \in S(f)$ is called an essential solution iff for any $\varepsilon > 0$, there exists $\delta > 0$ such that for any $f' \in Y$ with $\rho(f', f) < \delta$, there is $u' \in S(f')$ with $\|u - u'\| < \varepsilon$. (2) The optimal control problem associated f is called essential iff its solutions are all essential.

Theorem 3.3
$S : Y \rightrightarrows U$ is lower semicontinuous at $f \in Y$ if and only if the optimal control problem associated with f is essential.

Proof
Let $u \in S(f)$. For any $\varepsilon > 0$, the open neighborhood $N(u, \varepsilon)$ of u satisfies $N(u, \varepsilon) \cap S(f) \neq \emptyset$. If S is lower semicontinuous at f, then there exists $\delta > 0$ such that $N(u, \varepsilon) \cap S(f') \neq \emptyset$ for any $f' \in Y$ with $\rho(f', f) < \delta$. Take $u' \in N(u, \varepsilon) \cap S(f')$ and then $u' \in S(f')$ and $\|u - u'\| < \varepsilon$. Hence, the solution u is essential, and so the optimal control problem associated f is essential.

Conversely, if the optimal control problem associated $f \in Y$ is essential, then each of its solutions is essential. For any open set G with $G \cap S(f) \neq \emptyset$, there exists $u \in G \cap S(f)$. Then, G is an open neighborhood of u. There exists $\varepsilon > 0$ such that the open neighborhood $N(u, \varepsilon)$ of u satisfies $N(u, \varepsilon) \subset G$. Because u is an essential solution, there exists $\delta > 0$ such that for any $f' \in Y$ with $\rho(f', f) < \delta$, there is $u' \in S(f')$ with $\|u - u'\| < \varepsilon$. This shows $u' \in N(u, \varepsilon) \cap S(f') \subset G \cap S(f')$. Hence, $G \cap S(f') \neq \emptyset$ for any $f' \in Y$ with $\rho(f', f) < \delta$. Therefore, S is lower semicontinuous at f. □

Remark 3.2
If the optimal control problem associated $f \in Y$ is essential, by Theorems 3.2 and 3.3, the set-valued mapping $S : Y \rightrightarrows U$ is continuous at f. Because $S : Y \rightrightarrows U$ is compact valued, by Theorem 17.15 of [17], the solution set $S(f)$ is stable, that is, for any $\varepsilon > 0$, there exists $\delta > 0$ such that $H(S(f'), S(f)) < \delta$ when $\rho(f', f) < \varepsilon$, where H is the Hausdorff metric induced by the metric on U.

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Theorem 3.4
Let $f \in Y$. If $S(f) = \{u\}$ (i.e., $S(f)$ is a singleton), then the optimal control problem associated f is essential, and $u^k \to u$ for any $\{f^k\} \subset Y$ with $f^k \to f$ and any $u^k \in S(f^k)$.

Proof
Let $S(f) = \{u\}$. For any open neighborhood N of u in U, there is $N \supset S(f)$. By Theorem 3.2, S is upper semicontinuous at f. Then, there exists $\delta > 0$ such that $N \supset S(f')$ for any $f' \in Y$ with $\rho(f', f) < \delta$, of course, $N \cap S(f') \neq \emptyset$. This shows that S is lower semicontinuous at f. By Theorem 3.3, the optimal control problem associated f is essential.

In what follows, we will prove that if $\{f^k\} \subset Y$ with $f^k \to f$ and $u^k \in S(f^k)$, then $u^k \to u$. If it is not true, there must exist an open neighborhood O of u and a subsequence $\{u^{k_j}\}$ such that $u^{k_j} \notin O$. Because $u^{k_j} \in S\left(f^{k_j}\right) \subset U$ and U is compact, without loss of generality, we assume that $u^{k_j} \to u^* \in U$. Then, we have $u^* \notin O$. Because $f^{k_j} \to f$, $u^{k_j} \to u^*$ and Graph(S) is closed as proved in Theorem 3.2, we have $u^* \in S(f)$. It follows from $S(f) = \{u\}$ that $u^* = u \in O$, which contradicts $u^* \notin O$. This completes the proof. □

Theorem 3.5
There exists a dense residual subset Q of Y such that each optimal control problem associated $f \in Q$ is essential, and so every optimal control problem associated $f \in Y$ can be closely approximated arbitrarily by an essential optimal control problem.

Proof
By Theorem 3.2, $S : Y \rightrightarrows U$ is an usco mapping. Because (Y, ρ) is a complete metric space, by Lemma 3.2, there exists a dense residual subset Q of Y such that S is lower semicontinuous at each $f \in Q$. And so by Theorem 3.3, every optimal control problem associated $f \in Q$ is essential. Because Q is dense in Y, every optimal control problem associated $f \in Y$ can be closely approximated arbitrarily by an essential optimal control problem. □

Remark 3.3
Because Q is a second category set, we may say that, in the sense of Baire category, for most $f \in Y$, the solution set $S(f)$ of the optimal control problem associated f is stable.

Maybe one wonders whether for each $f \in Y$, the solution set $S(f)$ of the optimal control problem associated f is stable. The following example tells us it is not true.

Example 3.2
Let

$$U = \left\{u^k : u^k(t) = -\frac{2}{k}t + \frac{1}{k},\ t \in [0,1]; k = 1, 2, \cdots\right\}$$
$$\bigcup \{u^c : u^c(t) \equiv c,\ t \in [0,1]; c \in [-1, 0]\}.$$

Then, U is compact in $C[0,1]$. We aim to find an optimal control $u^* \in U$ such that

$$\begin{cases} \dot{x} = f(t, x, u^*),\ t \in [0,1], \\ x(0) = 0, \end{cases}$$

and

$$J_f(u^*) = \min_{u \in U} J_f(u)$$

where $J_f(u) = [x(1)]^2$.

(1) Take $f(t, x, u) = u(t), t \in [0,1]$. Then,

$$x(t) = A_f(u) = \begin{cases} -\frac{1}{k}t^2 + \frac{1}{k}t, & \text{if } u = u^k, k = 1, 2, \cdots \\ ct, & \text{if } u = u^c, c \in [-1, 0], \end{cases} \quad t \in [0,1],$$

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and

$$J_f(u) = [x(1)]^2 = \begin{cases} 0, & \text{if } u = u^k, k = 1, 2, \cdots \\ c^2, & \text{if } u = u^c, c \in [-1, 0]. \end{cases}$$

Hence,

$$S(f) = \arg\min_{u \in U} J_f(u) = \{u^k,\ k = 0, 1, 2, \cdots\} \subset U,$$

where for $k = 0$, it refers to $c = 0$, that is, $u^0(t) \equiv 0$.

(2) Take $f^m(t, x, u) = u(t) + \frac{1}{m}, t \in [0, 1]$, for each $m = 1, 2, \cdots$. Then,

$$x(t) = A_{f^m}(u) = \begin{cases} -\frac{1}{k}t^2 + \left(\frac{1}{k} + \frac{1}{m}\right)t, & \text{if } u = u^k, k = 1, 2, \cdots \\ \left(c + \frac{1}{m}\right)t, & \text{if } u = u^c, c \in [-1, 0], \end{cases} \quad t \in [0, 1],$$

and

$$J_{f^m}(u) = [x(1)]^2 = \begin{cases} \frac{1}{m^2}, & \text{if } u = u^k, k = 1, 2, \cdots \\ \left(c + \frac{1}{m}\right)^2, & \text{if } u = u^c, c \in [-1, 0]. \end{cases}$$

Hence,

$$S(f^m) = \arg\min_{u \in U} J_{f^m}(u) = \left\{u^{-\frac{1}{m}}\right\} \subset U.$$

(3) Obviously, we have $\rho(f^m, f) = \frac{1}{m} \to 0$, as $m \to \infty$. However, all solutions in $S(f)$ except u^0 are not essential, so f is not essential. In fact, for any $u^k \in S(f),\ k = 1, 2, \cdots$,

$$\|u^k - u^{-\frac{1}{m}}\| = \max_{t \in [0,1]} |-\frac{2}{k}t + \frac{1}{k} - \left(-\frac{1}{m}\right)| = \frac{1}{m} + \frac{1}{k} \to \frac{1}{k} > 0, \text{ as } m \to \infty,$$

which shows that u^k is not essential. We can also see that the Hausdorff distance

$$H(S(f^m), S(f)) = 1 + \frac{1}{m} \not\to 0, \text{ as } m \to \infty,$$

which implies that S is not lower semicontinuous at f and so f is not essential.

This example shows that although there may be infinite many optimal controllers for the given right-hand function f, only one optimal controller is stable with respect to the right-hand function f in the sense that when the right-hand function f perturbs slightly, the optimal controller does not perturb largely. This example also shows that the function f determines an unstable optimal control problems because the set $S(f)$ of all optimal controllers is not stable.

4. CONCLUSIONS

We know from the aforementioned example that not all the optimal control problems in Y are stable. Then, how many problems are not stable? Or how many problems are stable? The answer given by Theorem 3.5 is that the unstable (nonessential) optimal control problems in Y are only an 'exception', namely, a meagre set, and the stable (essential) optimal control problems in Y are 'overwhelming'. Although this paper analyzes the existence and stability of optimal solutions of optimal control problems with respect to only the right-hand side functions f, we will emphasize that most of the results can easily be generalized to the case that the functions g and h are also perturbed, that is, sequences $f_k \to f$, $g_k \to g$ and $h_k \to h$ can be considered. However, it is a regret that the presented result critically relies on the specific choice of the space U of admissible controls, which are assumed to be uniformly bounded and equicontinuous. This rules out the discontinuous controls. We will investigate whether the main result is still valid for more general cases in the future.

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