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Generic uniqueness theorems with some applications

Dingtao Peng · Jian Yu · Naihua Xiu

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Abstract We first present a proper condition under which the image of a set-valued mapping becomes a singleton and then obtain several generic uniqueness theorems which can be applied to study the uniqueness of the solutions for nonlinear problems. As applications, we prove that, in the sense of Baire category, most optimization problems (respectively, saddle point problems and variational inequality problems) have unique solution.

Keywords Generic uniqueness · Set-valued mapping · Optimization problem · Saddle point · Variational inequality problem

Mathematics Subject Classification 47H04 · 49K40 · 58E35

1 Introduction

Generic uniqueness of the solutions for various types of nonlinear problems has attracted the attention of many researchers and many results have been achieved. For example, Kenderov [7] studied the solutions of optimization problems and obtained an important result: most optimization problems have unique solution. Beer [2] generalized the results of [7] to constrained minimization problems. In [8], Kenderov and Ribarska proved that most two-person zero-sum continuous games have unique solution. Yu [11] investigated the stability of the weak efficient solutions for multi-objective optimization problems and included the main result of [2] as a special case. Tan, Yu and Yuan [10] studied the saddle points for general functions and derived the generic uniqueness of saddle points, which generalized the results in [8]. Recently, Zaslavski [14] proved that most minimization problems defined on a complete metric space have unique solution and that most of a class of equilibrium problems

D. Peng (✉) · N. Xiu
Department of Mathematics, School of Science, Beijing Jiaotong University, Beijing 100044, China
e-mail: dingtaopeng@126.com

D. Peng · J. Yu
Department of Mathematics, College of Science, Guizhou University, Guiyang 550025, Guizhou, China

have unique solution. By using the way similar to [10] but distinct from [14], Yu, Peng and Xiang [12] obtained the generic uniqueness for another class of equilibrium problems.

In this paper, we first present a proper condition under which the image of a set-valued mapping becomes a singleton and then obtain several generic uniqueness theorems. Next, we explain how do these generic uniqueness theorems work to study the uniqueness of the solutions for nonlinear problems. As applications, we prove that, in the sense of Baire category, most optimization problems (respectively, saddle point problems and variational inequality problems) have unique solution.

2 Preliminaries

First, let us recall some definitions and lemmas about the set-valued mappings, for more details to see [1].

Definition 2.1 [1,2] Let X, M be two topological spaces. We denote by 2^X the space of all nonempty subsets of X. Let $S : M \to 2^X$ be a set-valued mapping. Then (i) S is said to be upper (respectively, lower) semi-continuous at $u \in M$ if for each open set G in X with $G \supset S(u)$ (respectively, $G \cap S(u) \neq \emptyset$), there exists an open neighborhood O of u such that $G \supset S(u')$ (respectively, $G \cap S(u') \neq \emptyset$) for each $u' \in O$; (ii) S is said to be continuous at $u \in M$ if S is both upper semi-continuous and lower semi-continuous at u ; (iii) S is said to be an usco mapping if S is upper semi-continuous on M and $S(u)$ is compact for each $u \in M$; (iv) S is said to be almost lower semi-continuous at $u \in M$ if there exists $x \in S(u)$ such that for each open neighborhood U of x, there exists an open neighborhood O of u with the property that $U \cap S(u') \neq \emptyset$ for each $u' \in O$; (v) S is said to be closed if $\mathrm{Graph}(S)$ is closed, where $\mathrm{Graph}(S) := \{(u, x) \in M \times X : x \in S(u)\}$ is the graph of S.

Evidently, if $S : M \to 2^X$ is lower semi-continuous at u, it must be almost lower semi-continuous at u, but the converse is not true.

Lemma 2.1 [1] *If $S : M \to 2^X$ is closed and X is compact, then S is upper semi-continuous at each $u \in M$.*

Generally speaking, the upper semi-continuity and the lower semi-continuity are very different, of course none of them could include the other one, but Fort [6] provided the following lemma.

Lemma 2.2 [6] *Let M be a Hausdorff topological space, X be a metric space and $S : M \to 2^X$ be an usco mapping. Then there exists a residual subset Q of M (i.e., Q contains the intersection of a countable sequence of dense and open subsets of M) such that $\forall u \in Q$, S is lower semi-continuous at u.*

Note that if M is a Baire space, the residual subset of M must be dense in M. So Tan, Yu and Yuan [9] improved Lemma 2.2 as follows.

Lemma 2.3 [9] *Let M be a Baire space, X be a metric space and $S : M \to 2^X$ be an usco mapping. Then there exists a dense residual subset Q of M such that $\forall u \in Q$, S is lower semi-continuous at u.*

Remark 2.1 If there exists a dense residual subset Q of the Baire space M such that $\forall u \in Q$, a certain property P depending on u holds, then we say that the property P is generic on M.

Since Q is a second category set, we may say that the property P holds for most of the points (in the sense of Baire category) of M. In addition, since Q is dense in M, every $u \in M$ at which property P fails can be approached arbitrarily by a net in Q. The research on generic properties (including generic existence, generic uniqueness, generic stability, generic well-posedness and so on) has attracted much attention, see, for example, our references and the references given therein.

Definition 2.2 [2,7] M is called a Čech-complete space if it may be embedded as a residual subset of some compact Hausdorff space.

Definition 2.3 [2,7] A Hausdorff space X is said to belong to the class $\mathcal{L}$ if for every Čech-complete space M, every usco mapping $S : M \to 2^X$ is almost lower semi-continuous on some dense residual subset of M.

Remark 2.2 Each Čech-complete space is a Baire space; each locally compact Hausdorff space is Čech-complete; each completely metrizable space is Čech-complete; each metric space belongs to the class $\mathcal{L}$ (Lemma 2.3); and each Banach space with its weak topology (it is nonmetrizable) belongs to the class $\mathcal{L}$ (Theorem 2 of [5]). For more details to see [2,7,11] and the references given therein.

3 The condition for the image of a set-valued mapping to be a singleton and several generic uniqueness theorems

Let X, M be two Hausdorff topological spaces and $S : M \to 2^X$ be a set-valued mapping, $u \in M$.

We give a key condition denoted by "condition (C)" as follows:

For any two open sets G_1, G_2 in X with $G_1 \cap G_2 = \emptyset$, $S(u) \cap G_1 \neq \emptyset$ and any open neighborhood O of u, there exists $u' \in O \backslash \{u\}$ such that $S(u') \cap G_2 = \emptyset$.

One will see that for the image $S(u)$ to be a singleton, condition (C) sometimes is sufficient, sometimes is necessary, and sometimes even is sufficient and necessary.

Theorem 3.1 *Suppose that $S : M \to 2^X$ is upper semi-continuous at u. If $S(u)$ is a singleton, then condition (C) holds at u.*

Proof Let $S(u)$ be a singleton. For any open set G_1 in X with $S(u) \cap G_1 \neq \emptyset$, we have $S(u) \subset G_1$. Since S is upper semi-continuous at u, there exists an open neighborhood O_1 of u such that $S(u') \subset G_1$ for each $u' \in O_1$. For any open neighborhood O of u, it must hold that $O \cap O_1 \neq \emptyset$. Let $u' \in (O \cap O_1)\backslash\{u\}$, then $u' \in O\backslash\{u\}$ and $S(u') \subset G_1$. Hence for any open set G_2 in X with $G_1 \cap G_2 = \emptyset$, we have $S(u') \cap G_2 = \emptyset$. Thus condition (C) holds at u. □

Theorem 3.2 *Suppose that $S : M \to 2^X$ is almost lower semi-continuous at u. If condition (C) holds at u, then $S(u)$ is a singleton.*

Proof Since S is almost lower semi-continuous at u, there exists $\bar{x}$, tPr

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On one hand, since G_2 is an open neighborhood of $\bar{x}$, there exists an open neighborhood $O(u)$ of u such that

$$S(u') \cap G_2 \neq \emptyset \quad \text{for each } u' \in O(u). \tag{3.2}$$

On the other hand, since $G_1 \cap G_2 = \emptyset$, $S(u) \cap G_1 \neq \emptyset$ and condition (C) holds at u, for the previous open neighborhood $O(u)$, there exists $u' \in O(u)$ such that

$$S(u') \cap G_2 = \emptyset,$$

which is in contradiction with (3.2). Therefore $S(u)$ must be a singleton. □

Corollary 3.1 *Suppose that $S : M \to 2^X$ is lower semi-continuous at u. If condition (C) holds at u, then $S(u)$ is a singleton.*

Combining Theorems 3.1 and 3.2, we can get Theorem 3.3.

Theorem 3.3 *Suppose that $S : M \to 2^X$ is upper semi-continuous and almost lower semi-continuous at u, then $S(u)$ is a singleton if and only if condition (C) holds at u.*

Corollary 3.2 *Suppose that $S : M \to 2^X$ is continuous at u, then $S(u)$ is a singleton if and only if condition (C) holds at u.*

Furthermore, combining Lemma 2.3 and Corollary 3.2, we can get Theorem 3.4.

Theorem 3.4 *Let M be a Baire space, X be a metric space and $S : M \to 2^X$ be an usco mapping. If condition (C) holds at each point of M, then there exists a dense residual subset Q of M such that $\forall u \in Q$, $S(u)$ is a singleton.*

Corollary 3.3 *Let M be a complete metric space, X be a metric space and $S : M \to 2^X$ be an usco mapping. If condition (C) holds at each point of M, then there exists a dense residual subset Q of M such that $\forall u \in Q$, $S(u)$ is a singleton.*

From Theorem 3.2, Definitions 2.2 and 2.3, we derive the following theorem:

Theorem 3.5 *Let M be a Čech-complete space, X belong to the class $\mathcal{L}$ and $S : M \to 2^X$ be an usco mapping. If condition (C) holds at each point of M, then there exists a dense residual subset Q of M such that $\forall u \in Q$, $S(u)$ is a singleton.*

Now, we emphasize that the above results provide a unitary approach to investigate the uniqueness of the solutions for nonlinear problems. Concretely, let M denote the space of "problems". Each element $u \in M$ denotes a problem to be considered. $S(u)$ denotes the solution set of problem u, which is a subset of X. If we can show the (almost) lower semi-continuity of the solution mapping S at u and condition (C) holding at u, then we can derive the uniqueness of the solution for the problem u.

4 Applications

In this section, we will give three examples to demonstrate the application of our results from above.

4.1 Generic uniqueness of the solutions for optimization problems

Let X be a nonempty compact subset of a Hausdorff topological space E. The space M_1 of optimization problems is defined by

$$M_1 = \left\{ f : X \to R : \begin{array}{l} f \text{ is lower semi-continuous on } X \\ \text{and } \sup_{x \in X} |f(x)| < +\infty. \end{array} \right\}$$

Define a metric by

$$\rho_1(f_1, f_2) = \sup_{x \in X} |f_1(x) - f_2(x)|$$

for all $f_1, f_2 \in M_1$. Obviously, (M_1, ρ_1) is a complete metric space.

For each $f \in M_1$, the function f yields an optimization problem $\min_{u \in X} f(u)$. Denote its solution set by $S_1(f) := \{x \in X : f(x) = \min_{u \in X} f(u)\}$. Since X is compact and f is lower semi-continuous on X, $S_1(f) \neq \emptyset$. Then the correspondence $f \to S_1(f)$ gives a set-valued mapping.

Lemma 4.1 *$S_1 : M_1 \to 2^X$ is an usco mapping.*

Proof Since X is compact, by Lemma 2.1, it suffices to show that Graph(S_1) is a closed set, where Graph$(S_1) = \{(f, x) \in M_1 \times X : x \in S_1(f)\}$. Suppose that $\{f_\alpha\}_{\alpha \in \Lambda} \subset M_1$ with $f_\alpha \to f_0 \in M_1$ and $x_\alpha \in S_1(f_\alpha)$ with $x_\alpha \to x_0 \in X$, let us show $x_0 \in S_1(f_0)$.

$\forall u \in X, \forall \alpha \in \Lambda$, it follows from $x_\alpha \in S_1(f_\alpha)$ that $f_\alpha(x_\alpha) \leq f_\alpha(u)$. Since $f_\alpha \to f_0$, $x_\alpha \to x_0$ and $f_0(\cdot)$ is lower semi-continuous at x_0, we have

$$\begin{aligned} f_0(x_0) &\leq \liminf_\alpha f_0(x_\alpha) = \liminf_\alpha \left[\big(f_0(x_\alpha) - f_\alpha(x_\alpha)\big) + f_\alpha(x_\alpha)\right] \\ &\leq \lim_\alpha \rho_1(f_\alpha, f_0) + \liminf_\alpha f_\alpha(x_\alpha) \leq \lim_\alpha f_\alpha(u) = f_0(u). \end{aligned}$$

Therefore $f_0(x_0) = \min_{u \in X} f_0(u)$, $x_0 \in S_1(f_0)$ and thus the proof is complete. □

Lemma 4.2 *Condition (C) holds at every $f \in M_1$, that is, for any two open sets $G_1, G_2 \subset X$ with $G_1 \cap G_2 = \emptyset$, $S_1(f) \cap G_1 \neq \emptyset$ and any open neighborhood $O \subset M_1$ of f, there exists $f' \in O \setminus \{f\}$ such that $S_1(f') \cap G_2 = \emptyset$.*

Proof Suppose that $G_1, G_2 \subset X$ are two open sets with $G_1 \cap G_2 = \emptyset$ and $S_1(f) \cap G_1 \neq \emptyset$.

Take $x_1 \in S_1(f) \cap G_1$. Since X is a compact Hausdorff space, it is completely regular. Hence there exists a continuous function $g : X \to [0, 1]$ such that $g(x_1) = 0$ and $g(x) = 1$ for any $x \in X \setminus G_1$.

For each $n = 1, 2, \ldots$, define $f_n : X \to R$ as follows

$$f_n(x) := f(x) + \frac{1}{n} g(x), \quad \forall x \in X.$$

One can easily check that $\{f_n\} \subset M_1$ and $\rho_1(f_n, f) \leq \frac{1}{n} \to 0$ as $n \to \infty$. Next, we will prove that for each $n = 1, 2, \ldots, S_1(f_n) \cap G_2 = \emptyset$.

Denote by $a := \min_{u \in X} f(u)$. Then for any $x \in X$,

$$f_n(x) \geq f(x) \geq a.$$

Note that

$$f_n(x_1) = f(x_1) + \frac{1}{n} g(x_1) = f(x_1) = a.$$

Hence

$$\min_{u \in X} f_n(u) = a.$$

For any $x \in G_2 \subset X \setminus G_1$, it holds that

$$f_n(x) = f(x) + \frac{1}{n} g(x) = f(x) + \frac{1}{n} \geq a + \frac{1}{n} > a,$$

which implies that $S_1(f_n) \cap G_2 = \emptyset$.

For any open neighborhood $O \subset M_1$ of f, since $\rho_1(f_n, f) \to 0$, there exists $f_{n_0} \in O \setminus \{f\}$. Take $f' := f_{n_0}$, then $S_1(f') \cap G_2 = S_1(f_{n_0}) \cap G_2 = \emptyset$. Thus condition (C) holds at f. □

Theorem 4.1 *If X belongs to the class $\mathcal{L}$, then there exists a dense residual subset Q_1 of M_1 such that $\forall f \in Q_1$, $S_1(f)$ is a singleton, that is, the optimization problem f has a unique solution.*

Proof Because M_1 is a complete metric space, it is a Čech-complete space. By Lemma 4.1, $S_1 : M_1 \to 2^X$ is an usco mapping. Since X belongs to the class $\mathcal{L}$, there exists a dense residual subset Q_1 of M_1 such that $\forall f \in Q_1$, S_1 is almost lower semi-continuous at f. Lemma 4.2 shows that condition (C) holds at f. It follows from Theorem 3.2 that $S_1(f)$ is a singleton. □

Corollary 4.1 *If X is a nonempty compact subset of a metric space E, then there exists a dense residual subset Q_1' of M_1 such that $\forall f \in Q_1'$, the optimization problem f has a unique solution.*

Remark 4.1 Beer [2] also discussed the generic uniqueness of the solutions for constrained optimization problems with continuous objective functions. Although we have replaced the continuity by lower semi-continuity, we do not consider the perturbation of the feasible sets. We also mention that Zaslavski [14] obtained not only a generic uniqueness of solutions of optimization problems but also a generic well-posedness of optimization problems, a property which is stronger than the generic uniqueness. The difference between his generic uniqueness result and ours is that his minimization problems are defined on a complete metric space, while our minimization problems are defined on a compact Haudorff topological space belonging to the class $\mathcal{L}$. Also it is important that our approach is unitary, relying on checking the condition (C).

4.2 Generic uniqueness of the saddle points without convexity assumption

Let X, Y be two nonempty sets and $f : X \times Y \to R$ be a function. The so-called saddle point problem [10] is to find $(x^*, y^*) \in X \times Y$ such that

$$f(x, y^*) \leq f(x^*, y^*) \leq f(x^*, y), \quad \forall (x, y) \in X \times Y. \tag{4.1}$$

Then (x^*, y^*) is called a saddle point of f.

The saddle point has the following important and well-known property.

Lemma 4.3 *If $(x^*, y^*) \in X \times Y$ is a saddle point of f, then*

$$\max_{x \in X} \inf_{y \in Y} f(x, y) = \inf_{y \in Y} f(x^*, y) = f(x^*, y^*) = \sup_{x \in X} f(x, y^*) = \min_{y \in Y} \sup_{x \in X} f(x, y).$$

Let X and Y be nonempty compact and sequentially compact subsets of Hausdorff topological spaces E and F, respectively. The space M_2 of saddle point problems is defined by

$$M_2 = \left\{ f : X \times Y \to R : \begin{array}{l} \forall x \in X,\ \ y \to f(x,y) \ \text{ is lower semi-continuous on } Y, \\ \forall y \in Y,\ \ x \to f(x,y) \ \text{ is upper semi-continuous on } X, \\ \sup\limits_{(x,y)\in X\times Y} |f(x,y)| < +\infty, \\ \text{and } \exists (x^*, y^*) \in X \times Y \text{ such that (4.10) holds.} \end{array} \right\}$$

Define a metric by

$$\rho_2(f_1, f_2) = \sup_{(x,y)\in X\times Y} |f_1(x,y) - f_2(x,y)|$$

for all $f_1, f_2 \in M_2$.

Lemma 4.4 (M_2, ρ_2) *is a complete metric space.*

Proof Obviously, ρ_2 is a metric on M_2. We only need to show that (M_2, ρ_2) is complete. Let $\{f_n\}$ be a Cauchy sequence of M_2, then $\forall \epsilon > 0$, there exists a positive integer $N(\epsilon)$ such that $\forall m, n \geq N(\epsilon)$,

$$\rho_2(f_m, f_n) = \sup_{(x,y)\in X\times Y} |f_m(x,y) - f_n(x,y)| < \epsilon.$$

$\forall (x, y) \in X \times Y$, there exists $f(x, y) \in R$ such that $\lim\limits_{m\to\infty} f_m(x, y) = f(x, y)$ and $\forall n \geq N(\epsilon)$, we have

$$\sup_{(x,y)\in X\times Y} |f_n(x,y) - f(x,y)| \leq \epsilon. \tag{4.2}$$

Next, we prove $f \in M_2$.

One can easily prove that $\forall x \in X,\ \ y \to f(x, y)$ is lower semi-continuous on Y; $\forall y \in Y,\ \ x \to f(x, y)$ is upper semi-continuous on X and $\sup_{(x,y)\in X\times Y} |f(x, y)| < +\infty$.

Now we have to prove that the saddle point of f exists. For each $n = 1, 2, \ldots$, it follows from $f_n \in M_2$ that there exists $(x_n, y_n) \in X \times Y$ such that

$$f_n(x, y_n) \leq f_n(x_n, y_n) \leq f_n(x_n, y),\ \ \forall (x, y) \in X \times Y. \tag{4.3}$$

Since X and Y are both sequentially compact, without loss of generality, we assume that $x_n \to x^* \in X,\ \ y_n \to y^* \in Y$.

For any $(x, y) \in X \times Y$, by (4.2), for $n \geq N(\epsilon)$,

$$\begin{aligned} f(x, y^*) - f_n(x, y_n) &= f(x, y^*) - f(x, y_n) + f(x, y_n) - f_n(x, y_n) \\ &\leq f(x, y^*) - f(x, y_n) + \epsilon. \end{aligned} \tag{4.4}$$

Since $y \to f(x, y)$ is lower semi-continuous at y^* and $y_n \to y^*$, as well as (4.4), we have

$$\begin{aligned} f(x, y^*) - \liminf_{n\to\infty} f_n(x, y_n) &= \limsup_{n\to\infty} [f(x, y^*) - f_n(x, y_n)] \\ &\leq \limsup_{n\to\infty} [f(x, y^*) - f(x, y_n)] \\ &= f(x, y^*) - \liminf_{n\to\infty} f(x, y_n) \leq 0. \end{aligned} \tag{4.5}$$

On the other hand, by (4.2), for $n \geq N(\epsilon)$,

$$\begin{aligned} f_n(x_n, y) - f(x^*, y) &= f_n(x_n, y) - f(x_n, y) + f(x_n, y) - f(x^*, y) \\ &\leq f(x_n, y) - f(x^*, y) + \epsilon. \end{aligned} \tag{4.6}$$

Since $x \to f(x, y)$ is upper semi-continuous at x^* and $x_n \to x^*$, as well as (4.6), we have

$$\limsup_{n\to\infty} f_n(x_n, y) - f(x^*, y) \leq \limsup_{n\to\infty} f(x_n, y) - f(x^*, y) \leq 0. \tag{4.7}$$

It follows from (4.3), (4.5) and (4.7) that

$$\begin{aligned} f(x, y^*) &\leq \liminf_{n\to\infty} f_n(x, y_n) \leq \liminf_{n\to\infty} f_n(x_n, y_n) \\ &\leq \limsup_{n\to\infty} f_n(x_n, y_n) \leq \limsup_{n\to\infty} f_n(x_n, y) \leq f(x^*, y). \end{aligned} \tag{4.8}$$

Take $x = x^*$, $y = y^*$ in (4.8), then we obtain that $\lim_{n\to\infty} f_n(x_n, y_n)$ exists and $\lim_{n\to\infty} f_n(x_n, y_n) = f(x^*, y^*)$. Using (4.8) again, we get

$$f(x, y^*) \leq f(x^*, y^*) \leq f(x^*, y), \; \forall (x, y) \in X \times Y,$$

i.e., (4.1) holds. Thus $f \in M_2$. Consequently, inequality (4.2) implies $\lim_{n\to\infty} \rho_2(f_n, f) = 0$. Therefore (M_2, ρ_2) is complete. □

For every $f \in M_2$, the function f decides a saddle point problem. Denote by $S_2(f)$ the set of all saddle points of f. By the definition of M_2, $S_2(f) \neq \emptyset$, so that the correspondence $f \to S_2(f)$ gives a set-valued mapping.

Lemma 4.5 *$S_2 : M_2 \to 2^{X\times Y}$ is an usco mapping.*

Proof Since $X \times Y$ is compact in $E \times F$, by Lemma 2.1, it suffices to show that Graph(S_2) is closed, where Graph$(S_2) = \{(f, (x, y)) \in M_2 \times (X \times Y) : (x, y) \in S_2(f)\}$. Suppose that $\{f_\alpha\}_{\alpha\in\Lambda} \subset M_2$ with $f_\alpha \to f \in M_2$ and $(x_\alpha, y_\alpha) \in S_2(f_\alpha)$ with $x_\alpha \to x^* \in X$, $y_\alpha \to y^* \in Y$, let us show $(x^*, y^*) \in S_2(f)$.

$\forall (x, y) \in X \times Y$, $\forall \alpha \in \Lambda$, it follows from $(x_\alpha, y_\alpha) \in S_2(f_\alpha)$ that

$$f_\alpha(x, y_\alpha) \leq f_\alpha(x_\alpha, y_\alpha) \leq f_\alpha(x_\alpha, y).$$

Since $f_\alpha \to f$, $x_\alpha \to x^*$, $y_\alpha \to y^*$, through a proof analogous to that of Lemma 4.4, one can prove that

$$f(x, y^*) \leq f(x^*, y^*) \leq f(x^*, y),$$

which implies $(x^*, y^*) \in S_2(f)$. The proof is thus complete. □

Lemma 4.6 *Condition (C) holds at every $f \in M_2$, that is, for any two open sets $G_1, G_2 \subset X \times Y$ with $G_1 \cap G_2 = \emptyset$, $S_2(f) \cap G_1 \neq \emptyset$ and any open neighborhood $O \subset M_2$ of f, there exists $f' \in O\backslash\{f\}$ such that $S_2(f') \cap G_2 = \emptyset$.*

Proof Suppose that $G_1, G_2 \subset X \times Y$ are two open sets with $G_1 \cap G_2 = \emptyset$ and $S_2(f) \cap G_1 \neq \emptyset$. Take $(x_1, y_1) \in S_2(f) \cap G_1$. Then there exist two open sets $X_1 \subset X$, $Y_1 \subset Y$ such that $(x_1, y_1) \in X_1 \times Y_1 \subset G_1$.

Since X is a compact Hausdorff topological space, it is completely regular. Hence there exists a continuous function $g : X \to [0, 1]$ such that $g(x_1) = 0$ and $g(x) = 1$ for any

$x \in X \setminus X_1$. Similarly, since Y is a compact Hausdorff topological space, there also exists a continuous function $h : Y \to [0, 1]$ such that $h(y_1) = 0$ and $h(y) = 1$ for any $y \in Y \setminus Y_1$.

For each $n = 1, 2, \ldots$, define $f_n : X \times Y \to R$ as follows

$$f_n(x, y) := f(x, y) - \frac{1}{n} g(x)h(y),\ \forall (x, y) \in X \times Y.$$

One can easily check that $\forall x \in X,\ y \to f_n(x, y)$ is lower semi-continuous on Y; $\forall y \in Y,\ x \to f_n(x, y)$ is upper semi-continuous on X; and $\sup_{(x,y)\in X\times Y} |f_n(x, y)| < +\infty$. Since $(x_1, y_1) \in S_2(f)$, $g(x_1) = 0$ and $h(y_1) = 0$, for any $(x, y) \in X \times Y$, it holds that

$$\begin{aligned} f_n(x, y_1) &= f(x, y_1) - \frac{1}{n} g(x)h(y_1) = f(x, y_1) \le f(x_1, y_1) = f(x_1, y_1) - \frac{1}{n} g(x_1)h(y_1) \\ &= f_n(x_1, y_1) = f(x_1, y_1) \le f(x_1, y) = f(x_1, y) - \frac{1}{n} g(x_1)h(y) \\ &= f_n(x_1, y), \end{aligned}$$

which implies $(x_1, y_1) \in S_2(f_n)$. Thus $\{f_n\} \subset M_2$ and $\rho_2(f_n, f) \le \frac{1}{n} \to 0$ as $n \to \infty$. Next, we prove that $S_2(f_n) \cap G_2 = \emptyset$ for each $n = 1, 2, \ldots$.

Suppose, by contradiction, that there exist $n_0 > 0$ and $(x_{n_0}, y_{n_0}) \in X \times Y$ such that $(x_{n_0}, y_{n_0}) \in S_2(f_{n_0}) \cap G_2$. Since $G_1 \cap G_2 = \emptyset$, we have either $x_{n_0} \in X \setminus X_1$ or $y_{n_0} \in Y \setminus Y_1$. Without loss of generality, we assume $x_{n_0} \in X \setminus X_1$. Then $g(x_{n_0}) = 1$ and $(x_{n_0}, y_{n_0}) \in S_2(f_{n_0})$. Denote by

$$v_{n_0} := \max_{x\in X} \min_{y\in Y} f_{n_0}(x, y) \quad \text{and} \quad v_0 := \max_{x\in X} \min_{y\in Y} f(x, y).$$

(Note that, since X and Y are both compact and $f_{n_0}, f \in M_2$, "inf" and "sup" in Lemma 4.3 can be replaced by "min" and "max", respectively.) By Lemma 4.3, we have

$$\begin{aligned} v_{n_0} &= \max_{x\in X} \min_{y\in Y} f_{n_0}(x, y) = \min_{y\in Y} f_{n_0}(x_{n_0}, y) = \min_{y\in Y} \left[f(x_{n_0}, y) - \frac{1}{n_0} g(x_{n_0})h(y) \right] \\ &= \min_{y\in Y} f(x_{n_0}, y) - \frac{1}{n_0} \le \max_{x\in X} \min_{y\in Y} f(x, y) - \frac{1}{n_0} = v_0 - \frac{1}{n_0}. \end{aligned} \tag{4.9}$$

On the other hand, by $(x_1, y_1) \in S_2(f)$ and $g(x_1) = 0$, we have

$$\begin{aligned} v_0 &= \max_{x\in X} \min_{y\in Y} f(x, y) = \min_{y\in Y} f(x_1, y) = \min_{y\in Y} \left[f(x_1, y) - \frac{1}{n_0} g(x_1)h(y) \right] \\ &= \min_{y\in Y} f_{n_0}(x_1, y) \le \max_{x\in X} \min_{y\in Y} f_{n_0}(x, y) = v_{n_0}. \end{aligned} \tag{4.10}$$

Combining (4.9) and (4.10), we get

$$v_0 \le v_{n_0} \le v_0 - \frac{1}{n_0},$$

which is a contradiction. Thus we have shown that $S_2(f_n) \cap G_2 = \emptyset$ for each $n = 1, 2, \ldots$.

For any open neighborhood $O \subset M_2$ of f, since $\rho_2(f_n, f) \to 0$, there exists $f_{m_0} \in O \setminus \{f\}$. Take $f' := f_{m_0}$, then $S_2(f') \cap G_2 = S_2(f_{m_0}) \cap G_2 = \emptyset$. Thus condition (C) holds at f. □

Theorem 4.2 *If X and Y belong to the class $\mathcal{L}$, then there exists a dense residual subset Q_2 of M_2 such that $\forall f \in Q_2$, $S_2(f)$ is a singleton, that is, f has a unique saddle point.*

Proof Since M_2 is a complete metric space, it is a Čech-complete space. By Lemma 4.5, $S_2 : M_2 \to 2^{X\times Y}$ is an usco mapping. If X and Y belong to the class $\mathcal{L}$, by Proposition 2 (d) of [7], $X \times Y$ belongs to the class $\mathcal{L}$. Then there exists a dense residual subset Q_2 of M_2 such that $\forall f \in Q_2$, S_2 is almost lower semi-continuous at f. Lemma 4.6 shows that condition (C) holds at f. It follows from Theorem 3.2 that $S_2(f)$ is a singleton. □

Corollary 4.2 *If X and Y are nonempty compact subsets of the metric spaces E and F, respectively, then there exists a dense residual subset Q_2' of M_2 such that $\forall f \in Q_2'$, f has a unique saddle point.*

Remark 4.2 Theorem 4.2 generalize Theorem 1 of [10] from the following two aspects: (i) we do not require the convexity and concavity of the function $f \in M_2$; (ii) we do not require the convexity and linear structure of the sets X and Y. It should be also mentioned that the generic existence of a saddle point in the game theory was already established by Zaslavski [13]. However, the main result in [13] is different from Theorem 4.2. Firstly, the feasible sets in [13] are two complete metric spaces, while ours are two compact and sequentially compact Haudorff topological spaces belonging to the class $\mathcal{L}$. Secondly, the functions in [13] are continuous at two variables, while ours are upper semi-continuous at the first variable and lower semi-continuous at the second variable. Thirdly, the functions in [13] do not have to have a saddle point, while ours must have at least one saddle point. Therefore the two results can not include each other. We still emphasize that our approach is unitary.

4.3 Generic uniqueness of the solutions for variational inequalities

Let X be a nonempty compact subset of a Hilbert space H and $T : X \to H$ be a vector-valued function. The inner product and norm in H are denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$, respectively. The so-called variational inequality (for short, VI) [4,3] is to find $x^* \in X$ such that

$$\langle T(x^*), y - x^*\rangle \geq 0, \ \forall y \in X.$$

Then x^* is called a solution of VI(T).

Definition 4.1 [3] A vector-valued function $T : X \to H$ is said to be monotone on X if for any $x, y \in X$, it holds that $\langle T(x) - T(y), x - y\rangle \geq 0$.

The space M_3 of variational inequalities is defined by

$$M_3 = \left\{ T : X \to H : \begin{array}{l} T \ \text{ is continuous on } X, \\ T \ \text{ is monotone on } X, \\ \text{and } \exists x^* \in X \text{ such that } \langle T(x^*), y - x^*\rangle \geq 0, \ \forall y \in X. \end{array} \right\}$$

Define a metric by

$$\rho_3(T_1, T_2) = \max_{x\in X} \|T_1(x) - T_2(x)\|$$

for all $T_1, T_2 \in M_3$.

Lemma 4.7 *(M_3, ρ_3) is a complete metric space.*

Proof Obviously, ρ_3 is a metric on M_3. We only need to show that (M_3, ρ_3) is complete. Let $\{T_n\}$ be a Cauchy sequence of M_3, then $\forall \epsilon > 0$, there exists a positive integer $N(\epsilon)$ such that $\forall m, n \geq N(\epsilon)$,

$$\rho_3(T_m, T_n) = \max_{x\in X} \|T_m(x) - T_n(x)\| < \epsilon.$$

Since H is complete, for every $x \in X$, there exists $T(x) \in H$ such that $\lim_{m\to\infty} T_m(x) = T(x)$ and $\forall n \geq N(\epsilon)$, we have

$$\max_{x\in X} \|T_n(x) - T(x)\| \leq \epsilon. \tag{4.11}$$

Next, we prove $T \in M_3$.

One can easily prove that T is continuous and monotone on X.

Now we have to prove that there exists $x^* \in X$ such that $\langle T(x^*), y - x^*\rangle \geq 0$ for all $y \in X$. For each $n = 1, 2, \ldots$, it follows from $T_n \in M_3$ that there exists $x_n \in X$ such that

$$\langle T_n(x_n), y - x_n\rangle \geq 0, \quad \forall y \in X. \tag{4.12}$$

Note that X is compact in the Hilbert space H, without loss of generality, we assume that $x_n \to x^* \in X$. Since T is continuous at x^* and $x_n \to x^*$, there exists $N_1 \geq N(\epsilon)$ such that $\forall n \geq N_1$

$$\|T(x_n) - T(x^*)\| < \epsilon.$$

From the above inequality, as well as (4.11), we derive that $\forall n \geq N_1$

$$\|T_n(x_n) - T(x^*)\| \leq \|T_n(x_n) - T(x_n)\| + \|T(x_n) - T(x^*)\| < 2\epsilon,$$

which implies $\lim_{n\to\infty} \|T_n(x_n) - T(x^*)\| = 0$. Let $n \to \infty$ in inequality (4.12), then we derive

$$\langle T(x^*), y - x^*\rangle \geq 0, \quad \forall y \in X.$$

Thus $T \in M_3$. Consequently, inequality (4.11) implies $\lim_{n\to\infty} \rho_3(T_n, T) = 0$. Therefore (M_3, ρ_3) is complete. □

For every $T \in M_3$, denote the solution set of VI(T) by $S_3(T) := \{x \in X : \langle T(x), y - x\rangle \geq 0, \ \forall y \in X\}$. By the definition of M_3, $S_3(T) \neq \emptyset$. Then the correspondence $T \to S_3(T)$ decides a set-valued mapping.

Lemma 4.8 *$S_3 : M_3 \to 2^X$ is an usco mapping.*

Proof Since X is compact in H, by Lemma 2.1, it suffices to show that Graph(S_3) is closed, where Graph$(S_3) = \{(T, x) \in M_3 \times X : x \in S_3(T)\}$. Suppose that $T_n \in M_3$ with $T_n \to T \in M_3$ and $x_n \in S_3(T_n)$ with $x_n \to x^* \in X$, let us show $x^* \in S_3(T)$.

$\forall y \in X, \forall n = 1, 2 \ldots$, it follows from $x_n \in S_3(T_n)$ that

$$\langle T_n(x_n), y - x_n\rangle \geq 0.$$

Since $T_n \to T$, $x_n \to x^*$, through a proof analogous to that of Lemma 4.7, one can prove that

$$\langle T(x^*), y - x^*\rangle \geq 0.$$

which implies $x^* \in S_3(T)$. The proof is thus complete. □

Lemma 4.9 *Condition (C) holds at every $T \in M_3$, that is, for any two open sets $G_1, G_2 \subset X$ with $G_1 \cap G_2 = \emptyset$, $S_3(T) \cap G_1 \neq \emptyset$ and any open neighborhood $O \subset M_3$ of T, there exists $T' \in O\backslash\{T\}$ such that $S_3(T') \cap G_2 = \emptyset$.*

Proof Suppose that $G_1, G_2 \subset X$ are two open sets with $G_1 \cap G_2 = \emptyset$, $S_3(T) \cap G_1 \neq \emptyset$. Take $x_1 \in S_3(T) \cap G_1$, then

$$x_1 \in G_1 \quad \text{and} \quad \langle T(x_1), y - x_1 \rangle \geq 0, \ \forall y \in X. \tag{4.13}$$

For each $n = 1, 2, \ldots$, define $T_n : X \to H$ as follows

$$T_n(x) := T(x) + \frac{1}{n}(x - x_1), \quad \forall x \in X. \tag{4.14}$$

Since X is compact, X is bounded. Denote by $c := \max_{x \in X} \|x\|$. One can easily check that T_n is continuous on X and $\max_{x \in X \times X} \|T_n(x)\| < +\infty$.

Since T is monotone, for every $x, y \in X$, we have

$$\langle T(x) - T(y), x - y \rangle \geq 0.$$

Consequently,

$$\langle T_n(x) - T_n(y), x - y \rangle = \langle T(x) - T(y), x - y \rangle + \frac{1}{n}\|x - y\|^2 \geq 0,$$

which shows that T_n is monotone on X. Moreover, by (4.13) and (4.14),

$$\langle T_n(x_1), y - x_1 \rangle = \langle T(x_1), y - x_1 \rangle \geq 0, \ \forall y \in X,$$

which implies $x_1 \in S_3(T_n)$. Thus $\{T_n\} \subset M_3$ and $\rho_3(T_n, T) \leq \frac{2c}{n} \to 0$ as $n \to \infty$. Next, we prove that $S_3(T_n) \cap G_2 = \emptyset$ for each $n = 1, 2, \ldots$.

Suppose, by contradiction, that there exist $n_0 > 0$ and $x_{n_0} \in X$ such that $x_{n_0} \in S_3(T_{n_0}) \cap G_2$. Then $x_{n_0} \in G_2$, $x_{n_0} \neq x_1$ and $x_{n_0} \in S_3(T_{n_0})$. It follows from $x_{n_0} \in S_3(T_{n_0})$ that

$$\langle T_{n_0}(x_{n_0}), y - x_{n_0} \rangle \geq 0, \ \forall y \in X.$$

Take $y = x_1$, then

$$0 \leq \langle T_{n_0}(x_{n_0}), x_1 - x_{n_0} \rangle = \langle T(x_{n_0}), x_1 - x_{n_0} \rangle - \frac{1}{n_0}\|x_1 - x_{n_0}\|^2,$$

which implies

$$\langle T(x_{n_0}), x_1 - x_{n_0} \rangle \geq \frac{1}{n_0}\|x_1 - x_{n_0}\|^2 > 0, \tag{4.15}$$

It follows from $x_1 \in S_3(T)$ that

$$\langle T(x_1), y - x_1 \rangle \geq 0, \quad \forall y \in X.$$

Take $y = x_{n_0}$, then

$$\langle T(x_1), x_{n_0} - x_1 \rangle \geq 0. \tag{4.16}$$

Combining (4.15) and (4.16), we get

$$\langle T(x_1) - T(x_{n_0}), x_1 - x_{n_0} \rangle = -\left[\langle T(x_{n_0}), x_1 - x_{n_0} \rangle + \langle T(x_1), x_{n_0} - x_1 \rangle\right] < 0. \tag{4.17}$$

But, it follows from the monotonicity of T that

$$\langle T(x_1) - T(x_{n_0}), x_1 - x_{n_0} \rangle \geq 0.$$

which is in contradiction with (4.17). Thus we have shown that $S_3(T_n) \cap G_2 = \emptyset$ for each $n = 1, 2, \ldots$.

For any open neighborhood $O \subset M_3$ of T, since $\rho_3(T_n, T) \to 0$, there exists $T_{m_0} \in O \setminus \{T\}$. Take $T' := T_{m_0}$, then $S_3(T') \cap G_2 = S_3(T_{m_0}) \cap G_2 = \emptyset$. Thus condition (C) holds at T. □

Theorem 4.3 *There exists a dense residual subset Q_3 of M_3 such that $\forall T \in Q_3$, $S_3(T)$ is a singleton, that is, VI(T) has a unique solution.*

Proof Since M_3 is a complete metric space, it is a Baire space. By Lemma 4.8, $S_3 : M_3 \to 2^X$ is an usco mapping. Besides, X is a metric subspace. By Lemma 2.3, there exists a dense residual subset Q_3 of M_3 such that $\forall T \in Q_3$, S_3 is lower semi-continuous at T. Lemma 4.9 shows that condition (C) holds at T. We derive from Corollary 3.1 that $S_3(T)$ is a singleton. □

5 Conclusion

Our condition (C) does describe the essential character for the image of a set-valued mapping to be a singleton. For semi-continuous set-valued mappings, it could hardly be improved. Besides the applications to optimization problems, saddle point problems and variational inequality problems, the unified approach can be applied to study more nonlinear problems, which will be our future work. And our another future work is to make a comparison between the approach to generic uniqueness of Zaslavski [14] and ours.

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References

1. Aliprantis, C.D., Border, K.C.: Infinite Dimensional Analysis. Springer, Berlin (1999)
2. Beer, G.: On a generic optimization theorem of Petar Kenderov. Nonlinear Anal. TMA **12**, 627–655 (1988)
3. Bianchi, M., Schaible, S.: Equilibrium problems under generalized convexity and generalized monotonicity. J. Glob. Optim. **30**, 121–134 (2004)
4. Blum, E., Oettli, W.: From optimization and variational inequalities to equilibrium problems. Math. Stud. **63**, 123–145 (1994)
5. Christensen, J.P.R.: Theorems of Namioka and Johnsonn type for upper semi-continuous and compact valued set-valued mappings. Proc. Am. Math. Soc. **86**, 649–655 (1982)
6. Fort, M.K.: Points of continuity of semi-continuous functions. Publ. Math. Debrecen **2**, 100–102 (1951)
7. Kenderov P.S.: Most of the optimization problems have unique solution. In: Brosowski, B., Deutsch, F. (eds), Proceedings, Oberwolfach on Parametric Optimization. Birkhäuser International Series of Numerical Mathematics, vol. 72, pp. 203–216. Birkhäuser, Basel (1984)
8. Kenderov, P.S., Ribarska, N.K.: Most of the two person zero-sum games have unique solution, Workshop/Mini-Conference on Functional Analysis and Optimization, Canberra, pp. 73–82 (1988)
9. Tan, K.K., Yu, J., Yuan, X.Z.: The stability of Ky Fan's points. Proc. Am. Math. Soc. **123**, 1511–1519 (1995)
10. Tan, K.K., Yu, J., Yuan, X.Z.: The uniqueness of saddle points. Bull. Pol. Acad. Sci. Math. **43**, 119–129 (1995)
11. Yu, J.: Essential weak effficient solution in multiobjective optimization problems. J. Math. Anal. Appl. **166**, 230–235 (1992)
12. Yu, J., Peng, D.T., Xiang, S.W.: Generic uniqueness of equilibrium points. Nonlinear Anal. TMA **74**, 6326–6332 (2011)
13. Zaslavski, A.J.: Generic existence of a saddle point. Commun. Appl. Anal. **8**, 143–151 (2004)
14. Zaslavski, A.J.: Optimization on Metric and Normed Spaces. Springer, New York (2010)