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Generic Uniqueness of Solutions for a Class of Vector Ky Fan Inequalities

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Abstract This paper is intended mainly to present some generic uniqueness results for a class of vector Ky Fan inequalities. By employing the methods of set-valued analysis, we prove that, in the sense of Baire category, most of the problems in a complete metric space, consisting of vector Ky Fan inequalities satisfying some conditions, have unique solution and that every vector Ky Fan inequality, possessing more than one solution, can be approached arbitrarily by a sequence of vector Ky Fan inequalities each of which has a unique solution. Our discussions are under two different settings. One setting is related to vector Ky Fan inequalities defined on a compact set; the other is related to vector Ky Fan inequalities defined on a noncompact set. The corollaries of our results generalized the corresponding results in the literature.

Keywords Vector Ky Fan inequality · Generic uniqueness · Set-valued mapping · Complete metric space · Dense residual set

1 Introduction

In [1], Ky Fan introduced an important inequality which now has been called Ky Fan inequality. Until now, such an inequality plays an important role in Variational Analysis (it embraces Stampacchia and Minty inequalities), and hence in optimization. This type of inequalities has been studied also in [2]. In 1995, Tan, Yu, and Yuan first named solutions of Ky Fan inequalities "Ky Fan's points", and then investigated the stability of Ky Fan's points (see [3]). Vector Ky Fan inequalities (some authors called

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vector equilibrium problems) are natural generalizations of Ky Fan inequalities to vector-valued functions. They contain many mathematical problems as special cases such as vector variational inequality, vector optimization, vector complementarity, multiobjective game, and so on. The study of vector Ky Fan inequalities has recently been a rapidly growing area of research. See, for example, [4–6] and the references therein. However, most of these works focus on the solvability of vector Ky Fan inequalities. The research on the uniqueness of solutions for vector Ky Fan inequalities was hardly seen.

On the other hand, as we all know, the uniqueness of solutions for mathematical problems is very important in theory, algorithm, and application. But, we must notice that, except for few mathematical problems, most of the mathematical problems cannot guarantee the uniqueness of the solution. Some natural questions will be asked. For example, when do they have unique solution? How many problems in a class of problems are there having unique solution? Thus, researchers began to investigate the generic uniqueness which will be introduced in Remark 2.1. About the generic uniqueness, many results have been achieved. For example, Kenderov studied the solutions of optimization problems and obtained an important result: most optimization problems (satisfying some conditions) have unique solution (see [7]). Beer generalized the results of [7] to Čech-complete spaces (see [8]). In [9], Kenderov and Ribarska proved that most two-person zero-sum continuous games have unique solution. Tan, Yu, and Yuan studied the uniqueness of saddle points for general functions and derived the generic uniqueness of saddle points (see [10]). Recently, Zaslavski proved the generic uniqueness for a class of equilibrium problems (see [11]); By using the way similar to [10] but distinct from [11], Yu, Peng, and Xiang obtained the generic uniqueness for another class of equilibrium problems (see [12]). More results about the generic uniqueness can be found in [13] by Zaslavski.

The main purpose of this paper is to present some generic uniqueness results for a class of vector Ky Fan inequalities. By employing the methods of set-valued analysis, we prove that, in the sense of Baire category, most of the problems in a complete metric space, consisting of vector Ky Fan inequalities satisfying some conditions, have unique solution and that every vector Ky Fan inequality, possessing more than one solution, can be approached arbitrarily by a sequence of vector Ky Fan inequalities each of which has a unique solution.

This paper is organized as follows. In Sect. 2, we recall some notions and preliminaries. In Sect. 3, we investigate the uniqueness of solutions for vector Ky Fan inequalities defined on a compact set. For this case, we consider the perturbation of vector-valued functions. Section 4 studies the uniqueness of solutions for vector Ky Fan inequalities defined on a noncompact set. In this section, we not only consider the perturbation of vector-valued functions, but also consider the perturbation of feasible sets. Section 5 gives some examples and remarks. At last, we conclude our results in Sect. 6.

2 Preliminaries

Throughout this section, let H denote a Hausdorff topological vector space and C denote a nonempty, closed, convex, and pointed cone in H with apex at the origin and $\operatorname{int} C \neq \emptyset$, where $\operatorname{int} C$ denotes the topological interior of C.

Let X be a nonempty set and $\phi : X \times X \to H$ be a vector-valued function. The vector Ky Fan inequality which we will deal with is to find $x^* \in X$ such that

$$\phi\big(x^*, y\big) \in C, \quad \forall y \in X. \tag{1}$$

We call x^* in (1) a solution of the vector Ky Fan inequality ϕ.

When $H = \mathbb{R}$ and $C =]-\infty, 0]$, the vector Ky Fan inequality becomes the Ky Fan inequality (see [1, 3]): to find $x^* \in X$ such that

$$\phi\big(x^*, y\big) \le 0, \quad \forall y \in X. \tag{2}$$

To investigate the uniqueness of solutions for vector Ky Fan inequalities, let us first recall some definitions and lemmas about set-valued mappings; for more details, see [14].

Definition 2.1 Let X, Y be two topological spaces and $F : X \rightrightarrows Y$ be a set-valued mapping.

(1) F is said to be upper (respectively, lower) semicontinuous at $x \in X$ iff for each open set G in Y with $G \supset F(x)$ (respectively, $G \cap F(x) \ne \emptyset$), there exists an open neighborhood $O(x) \subset X$ of x such that $G \supset F(x')$ (respectively, $G \cap F(x') \ne \emptyset$) for any $x' \in O(x)$;
(2) F is said to be continuous at $x \in X$ iff F is both upper semicontinuous and lower semicontinuous at x;
(3) F is said to be an usco mapping iff F is upper semicontinuous and $F(x)$ is nonempty compact for each $x \in X$.
(4) F is said to be closed iff $\mathrm{Graph}(F)$ is closed, where $\mathrm{Graph}(F) := \{(x, y) \in X \times Y : x \in X, y \in F(x)\}$ is the graph of F.

Definition 2.2 Let X be a topological spaces. A subset Q of X is called residual iff it contains a countable intersection of open dense subsets of X.

Lemma 2.1 *If $F : X \rightrightarrows Y$ is closed and Y is compact, then F is upper semicontinuous at each $x \in X$.*

Generally speaking, the upper semicontinuity and the lower semicontinuity are very different. Of course, none of them could include the other one, but we have the following Fort's theorem.

Theorem 2.1 (Fort [15]) *Let X be a Hausdorff topological space, Y be a metric space and $F : X \rightrightarrows Y$ be an usco mapping, then there exists a residual subset Q of X such that F is lower semicontinuous at each $x \in Q$.*

Note that if X is a Baire space, then the residual subset of X must be dense in X. So Fort's theorem can be stated as follows.

Theorem 2.2 (See [3]) *Let X be a Baire space, Y be a metric space and $F : X \rightrightarrows Y$ be an usco mapping, then there exists a dense residual subset Q of X such that F is lower semicontinuous at each $x \in Q$.*

Remark 2.1 If there exists a dense residual subset Q of X such that for each $x \in Q$, a certain property P depending on x holds, then we say that the property P is generic on X. Since Q is a second category set, we may say that the property P holds for most of the points (in the sense of Baire category) in X. The research on generic properties (including generic existence, generic uniqueness, generic stability, generic well-posedness, and so on) has attracted much attention, see [3, 7–13, 16, 17] and the references therein.

Next, we recall a definition about vector-valued functions; for more details, see [5].

Definition 2.3 Let X be a nonempty subset of a Hausdorff topological vector space E and $f : X \to H$ be a vector-valued function. Then f is said to be C-upper semicontinuous at $x \in X$ iff for any open neighborhood V of 0 in H, there exists an open neighborhood U of x in X such that for any $x' \in U$,

$$f(x') \in f(x) + V - C \quad \big(\text{or equivalently, } f(x) \in f(x') + V + C\big); \tag{3}$$

f is said to be C-upper semicontinuous on X iff f is C-upper semicontinuous at each $x \in X$; and f is said to be C-lower semicontinuous on X iff $-f$ is C-upper semicontinuous on X.

Example 2.1 Let $E = \mathbb{R}$, $X = [-1, 1] \subset E$, $H = \mathbb{R}^2$ and $C = \mathbb{R}^2_+ \subset H$. Set

$$f_1(x) := \begin{cases} (0,0)^\top, & -1 \le x < 0, \\ (1,1)^\top, & 0 \le x \le 1, \end{cases} \qquad f_2(x) := (x, x)^\top, \quad -1 \le x \le 1.$$

One can easily check that f_1 is C-upper semicontinuous on X and that f_2 is both C-upper semicontinuous on X and C-lower semicontinuous on X.

Lemma 2.2 (See [18]) $\operatorname{int} C + C \subset \operatorname{int} C$.

3 Generic Uniqueness in the Case of Compact Set

Throughout this section, let X be a nonempty compact subset of a metric space E, $(H, \|\cdot\|)$ be a Banach space and C be a nonempty, closed, convex, and pointed cone in H with apex at the origin and $\operatorname{int} C \neq \emptyset$. For any $\epsilon > 0$, denote by $B(\epsilon) := \{z \in H : \|z\| \le \epsilon\}$ and $B^\circ(\epsilon) := \{z \in H : \|z\| < \epsilon\}$. We emphasize that the open neighborhood V in Definition 2.3 can be replaced by $B^\circ(\epsilon)$ in the setting of H being a normed space.

The problem space M is given by

$$M := \left\{ \phi : X \times X \to H : \begin{array}{l} \forall y \in X, \\ \quad x \to \phi(x, y) \text{ is } C\text{-upper semicontinuous on } X, \\ \forall x, y \in X,\ x \neq y,\ \phi(x, y) + \phi(y, x) \notin \operatorname{int} C, \\ \sup\limits_{(x,y) \in X \times X} \|\phi(x, y)\| < +\infty \text{ and} \\ \exists x \in X \text{ such that } \phi(x, y) \in C,\ \forall y \in X \end{array} \right\}. \tag{4}$$

For each $\phi_1, \phi_2 \in M$, define

$$\rho_1(\phi_1, \phi_2) := \sup_{(x,y)\in X\times X} \left\|\phi_1(x, y) - \phi_2(x, y)\right\|.$$

Example 3.1 Let $E = H = \mathbb{R}^2$, $X = [0, 1] \times [0, 1] \subset E$, $C = \mathbb{R}^2_+ \subset H$, and the space M be defined by (4). Let $\phi(x, y) := x - y = (x_1 - y_1, x_2 - y_2)^\top, \forall x = (x_1, x_2)^\top$, $y = (y_1, y_2)^\top \in X$. One can easily check that $x_0 := (1, 1)^\top \in X$ is a solution of the vector Ky Fan inequality ϕ and that ϕ is an element of M.

Lemma 3.1 *(M, ρ_1) is a complete metric space.*

Proof Obviously, ρ_1 is a metric on M. We only need to show that (M, ρ_1) is complete. Let $\{\phi_n\}$ be a Cauchy sequence of M, then for any $\epsilon > 0$, there exists a positive integer $N(\epsilon)$ such that for any $m, n \geq N(\epsilon)$,

$$\rho_1(\phi_m, \phi_n) = \sup_{(x,y)\in X\times X} \left\|\phi_m(x, y) - \phi_n(x, y)\right\| < \epsilon.$$

For each $x, y \in X$, since H is a Banach space, there exists $\phi(x, y) \in H$ such that $\lim_{m\to\infty} \phi_m(x, y) = \phi(x, y)$ and for any $n \geq N(\epsilon)$,

$$\sup_{(x,y)\in X\times X} \left\|\phi_n(x, y) - \phi(x, y)\right\| \leq \epsilon. \tag{5}$$

Next, we prove $\phi \in M$.

(i) Fix $n \geq N(\epsilon)$. Since $x \to \phi_n(x, y)$ is C-upper semicontinuous, there exists a neighborhood $U(x) \subset X$ of x such that for any $x' \in U(x)$,

$$\phi_n\big(x', y\big) \in \phi_n(x, y) + B^\circ(\epsilon) - C. \tag{6}$$

Thus, by (5) and (6), it holds that for any $x' \in U(x)$,

$$\phi\big(x', y\big) \in \phi_n\big(x', y\big) + B(\epsilon) \subset \phi_n(x, y) + B^\circ(2\epsilon) - C \subset \phi(x, y) + B^\circ(3\epsilon) - C.$$

It follows that $x \to \phi(x, y)$ is C-upper semicontinuous.

(ii) For any $x, y \in X, x \neq y$, we have $\phi_n(x, y) + \phi_n(y, x) \notin \operatorname{int} C$ for each $n = 1, 2, \ldots$. Since $\operatorname{int} C$ is open, $\lim_{n\to\infty} \phi_n(x, y) = \phi(x, y)$ and $\lim_{n\to\infty} \phi_n(y, x) = \phi(y, x)$, it follows that $\phi(x, y) + \phi(y, x) \notin \operatorname{int} C$.

(iii) For each $n \geq N(\epsilon)$, we have

$$\sup_{(x,y)\in X\times X} \left\|\phi_n(x, y) - \phi(x, y)\right\| \leq \epsilon \quad \text{and} \quad \sup_{(x,y)\in X\times X} \left\|\phi_n(x, y)\right\| < +\infty.$$

Hence,

$$\sup_{(x,y)\in X\times X} \left\|\phi(x, y)\right\| \leq \sup_{(x,y)\in X\times X} \left\|\phi_n(x, y)\right\| + \epsilon < +\infty.$$

(iv) Since $\phi_n \in M$ for each $n = 1, 2, \ldots$, there exists $x_n \in X$ such that

$$\phi_n(x_n, y) \in C, \quad \forall y \in X. \tag{7}$$

Note that $\{x_n\} \subset X$ and X is compact in the metric space E. Without loss of generality, we may assume that $x_n \to x^* \in X$. For any $y \in X$, we have proved in (i) that

$x \to \phi(x, y)$ is C-upper semicontinuous on X. Hence, there exists $N_2 \geq N(\epsilon)$ such that for any $n \geq N_2$,

$$\phi(x^*, y) - \phi(x_n, y) \in B^\circ(\epsilon) + C. \tag{8}$$

From (5), we derive that for any $n \geq N_2$,

$$\phi(x_n, y) - \phi_n(x_n, y) \in B(\epsilon). \tag{9}$$

By (7)–(9), we obtain that for any $n \geq N_2$,

$$\begin{aligned} \phi(x^*, y) &= \big[\phi(x^*, y) - \phi(x_n, y)\big] + \big[\phi(x_n, y) - \phi_n(x_n, y)\big] + \phi_n(x_n, y) \\ &\in B^\circ(\epsilon) + B(\epsilon) + C + C \\ &\subset B^\circ(2\epsilon) + C. \end{aligned} \tag{10}$$

By the arbitrariness of $\epsilon > 0$, we derive from (10) that $\phi(x^*, y) \in C$ for all $y \in X$.

Thus, we have shown that $\phi \in M$. Consequently, inequality (5) implies that $\lim_{n\to\infty} \rho_1(\phi_n, \phi) = 0$. Therefore, (M, ρ_1) is a complete metric space. □

By the definition of M, for each $\phi \in M$, the vector Ky Fan inequality ϕ must have at least one solution, i.e., $\exists x \in X$ such that $\phi(x, y) \in C$ for all $y \in X$. Denote by $S_1(\phi)$ the set of all solutions of the vector Ky Fan inequality ϕ. Then the correspondence $\phi \to S_1(\phi)$ defines a set-valued mapping $S_1 : M \rightrightarrows X$.

Lemma 3.2 *$S_1 : M \rightrightarrows X$ is an usco mapping.*

Proof Since X is compact, by Lemma 2.1, it suffices to show that $\mathrm{Graph}(S_1)$ is closed, where $\mathrm{Graph}(S_1) = \{(\phi, x) \in (M, X) : x \in S_1(\phi)\}$. Let $\{\phi_n\} \subset M$ with $\phi_n \to \phi_0 \in M$ and $x_n \in S_1(\phi_n)$ with $x_n \to x_0$, then let us show $x_0 \in S_1(\phi_0)$.

For any $y \in X$, since $x_n \in S_1(\phi_n)$, we have

$$\phi_n(x_n, y) \in C. \tag{11}$$

Note that $\phi_0 \in M$, then $x \to \phi_0(x, y)$ is C-upper semicontinuous at x_0. Hence, $x_n \to x_0$ implies that there exists $N_1 > 0$ such that for any $n \geq N_1$,

$$\phi_0(x^*, y) - \phi_0(x_n, y) \in B^\circ\left(\frac{\epsilon}{2}\right) + C. \tag{12}$$

Since $\phi_n \to \phi_0$, there exists $N_2 \geq N_1$ such that for any $n \geq N_2$,

$$\phi_0(x_n, y) - \phi_n(x_n, y) \in B^\circ\left(\frac{\epsilon}{2}\right). \tag{13}$$

By (11)–(13), we obtain that for any $n \geq N_2$,

$$\begin{aligned} \phi_0(x_0, y) &= \big[\phi_0(x_0, y) - \phi_0(x_n, y)\big] + \big[\phi_0(x_n, y) - \phi_n(x_n, y)\big] + \phi_n(x_n, y) \\ &\in B^\circ\left(\frac{\epsilon}{2}\right) + B^\circ\left(\frac{\epsilon}{2}\right) + C + C \\ &\subset B^\circ(\epsilon) + C. \end{aligned} \tag{14}$$

By the arbitrariness of $\epsilon > 0$, we derive from (14) that $\phi_0(x_0, y) \in C$ for all $y \in X$, i.e., $x_0 \in S_1(\phi_0)$. Therefore, $S_1 : M \rightrightarrows X$ is an usco mapping. □

Theorem 3.1 *There exists a dense residual subset* Q_1 *of* M *such that* $S_1(\phi)$ *is a singleton for each* $\phi \in Q_1$.

Proof By Lemma 3.1, M is a complete metric space, hence it is a Baire space. By Lemma 3.2, $S_1 : M \rightrightarrows X$ is an usco mapping. Besides, X is a subspace of the metric space E. By Theorem 2.2, there exists a dense residual subset Q_1 of M such that S_1 is lower semicontinuous at each $\phi \in Q_1$.

By the way of contradiction, we assume that $S_1(\phi)$ is not a singleton for some $\phi \in Q_1$. Then there exist at least two points $x_1, x_2 \in S_1(\phi)$ with $x_1 \neq x_2$. Therefore, there exists an open neighborhood U of x_1 and an open neighborhood V of x_2 such that $U \cap V = \emptyset$.

Define a function $g : X \to \mathbb{R}$ as follows:

$$g(x) := \frac{d(x, x_1)}{d(x, x_1) + d(x, X \setminus U)}, \quad \forall x \in X.$$

where d is the metric on X. Note that $g : X \to \mathbb{R}$ is continuous on X; $\forall x \in X$, $0 \le g(x) \le 1$; $g(x_1) = 0$; $\forall x \in V$, $g(x) = 1$.

Take $z \in \operatorname{int} C$. For each $n = 1, 2, \ldots$, let $\phi_n : X \times X \to H$ be defined by

$$\phi_n(x, y) := \phi(x, y) + \frac{1}{n}\big[g(y) - g(x)\big]z, \quad \forall x, y \in X.$$

For each $n = 1, 2, \ldots$, one can check that:

(1) $\forall y \in X, x \to \phi_n(x, y)$ is C-upper semicontinuous on X;
(2) $\forall x, y \in X, x \neq y, \phi_n(x, y) + \phi_n(y, x) = \phi(x, y) + \phi(y, x) \notin \operatorname{int} C$;
(3) $\sup_{(x,y)\in X\times X} \|\phi_n(x, y)\| \le \sup_{(x,y)\in X\times X} \|\phi(x, y)\| + \frac{1}{n} \cdot \|z\| < +\infty$;
(4) $S_1(\phi_n) \neq \emptyset$, which proof is given as follows.

In fact, for any fixed $y \in X$, we have $\phi(x_1, y) \in C$ since $x_1 \in S_1(\phi)$. Note that $g(x_1) = 0$ and $\frac{1}{n} g(y)z \in C$. Then we can get that

$$\begin{aligned}\phi_n(x_1, y) &= \phi(x_1, y) + \frac{1}{n}\big[g(y) - g(x_1)\big]z \\ &= \phi(x_1, y) + \frac{1}{n} g(y)z \\ &\in C + C \subset C.\end{aligned}$$

It implies that $x_1 \in S_1(\phi_n)$. Hence, $S_1(\phi_n) \neq \emptyset$.

(5) $\rho_1(\phi_n, \phi) = \frac{1}{n} \cdot \|z\| \cdot \max_{(x,y)\in X\times X} |g(x) - g(y)| = \frac{1}{n}\|z\| \to 0$ as $n \to \infty$.

Clearly, we have shown that $\phi_n \in M$ for each $n = 1, 2, \ldots$ and $\phi_n \to \phi$ as $n \to \infty$.

Note that $x_2 \in V \cap S_1(\phi)$, then $V \cap S_1(\phi) \neq \emptyset$. Since S_1 is lower semicontinuous at ϕ and $\phi_n \to \phi$, there exists a positive integer n_0 big enough such that $V \cap S_1(\phi_{n_0}) \neq \emptyset$. Taking $x_{n_0} \in V \cap S_1(\phi_{n_0})$, then we have $x_{n_0} \in V$ and $\phi_{n_0}(x_{n_0}, y) \in C$ for all $y \in X$. Especially, taking $y = x_1$, we get

$$\phi_{n_0}(x_{n_0}, x_1) \in C. \tag{15}$$

Since $g(x_1)=0$ and $g(x_{n_0})=1$ because of $x_{n_0}\in V$, it follows that

$$\phi_{n_0}(x_{n_0},x_1)=\phi(x_{n_0},x_1)+\frac{1}{n_0}\big[g(x_1)-g(x_{n_0})\big]z=\phi(x_{n_0},x_1)-\frac{1}{n_0}z. \tag{16}$$

Noting that $\frac{1}{n_0}z\in \operatorname{int}C$, together with (15), (16), and Lemma 2.2, we get

$$\phi(x_{n_0},x_1)=\phi_{n_0}(x_{n_0},x_1)+\frac{1}{n_0}z\in C+\operatorname{int}C\subset \operatorname{int}C. \tag{17}$$

Since $x_1\in S_1(\phi)$, we have $\phi(x_1,y)\in C$ for all $y\in X$. Taking $y=x_{n_0}$, we get

$$\phi(x_1,x_{n_0})\in C. \tag{18}$$

It follows from (17), (18), and Lemma 2.2 that

$$\phi(x_{n_0},x_1)+\phi(x_1,x_{n_0})\in \operatorname{int}C+C\subset \operatorname{int}C. \tag{19}$$

But $\phi\in M$ and $x_1\neq x_{n_0}$ imply that

$$\phi(x_{n_0},x_1)+\phi(x_1,x_{n_0})\notin \operatorname{int}C,$$

which is in contradiction with (19). Therefore, $S_1(\phi)$ must be a singleton. □

When $H=\mathbb{R}$, $C=[0,+\infty[$, we get the following Corollary 3.1.

Corollary 3.1 *Let*

$$M':=\left\{ f:X\times X\to\mathbb{R}: \begin{array}{l} \forall y\in X,\ x\to f(x,y)\ \textit{is upper semicontinuous on } X,\\ \forall x,y\in X,\ x\neq y,\ f(x,y)+f(y,x)\le 0,\\ \sup\limits_{(x,y)\in X\times X}\big|f(x,y)\big|<+\infty\ \textit{and}\\ \exists x\in X\ \textit{such that } f(x,y)\ge 0,\ \forall y\in X \end{array}\right\}.$$

Then there exists a dense residual subset Q_1' of M' such that each $f\in Q_1'$ has a unique equilibrium point.

Remark 3.1 Corollary 3.1 generalized Theorem 2.2 of [12], one of main results of [12], from the following four aspects: (i) we do not require any convexity of the function $f\in M'$; (ii) we do not require any convexity or linear structure of the space X; (iii) we omit the requirement that $f(x,x)\ge 0$ for all $x\in X$; (iv) we do not require $f(x,y)+f(y,x)\le 0$ holding for $x=y$. For more details, see [12].

4 Generic Uniqueness in the Case of Noncompact Set

In Sect. 3, we discussed the uniqueness of the solutions for vector Ky Fan inequalities with the perturbation of the vector-valued functions defined on a compact set.

In this section, we shall not only consider the perturbation of the vector-valued functions, but also consider the perturbation of the feasible sets contained in a noncompact set.

Throughout this section, let X be a nonempty, bounded, and closed subset of a complete metric space E. Note that X is complete but could be noncompact. $(H, \|\cdot\|)$ is still a Banach space and C is a nonempty, closed, convex and pointed cone in H with apex at the origin and $\operatorname{int} C \neq \emptyset$. For any $\epsilon > 0$, $B^\circ(\epsilon)$ and $B(\epsilon)$ are the same as those defined in Sect. 3.

The space Y of problems is defined by

$$Y := \left\{ u = (\phi, A) : \begin{array}{l} \phi : X \times X \to H \text{ is } C\text{-upper semicontinuous on } X \times X, \\ \forall x, y \in X,\ x \neq y,\ \phi(x,y) + \phi(y,x) \notin \operatorname{int} C, \\ \sup_{(x,y)\in X\times X} \|\phi(x,y)\| < +\infty, \\ A \text{ is a nonempty compact subset of } X \text{ and} \\ \exists x \in A \text{ such that } \phi(x,y) \in C,\ \forall y \in A \end{array} \right\}.$$

For each $u_1 = (\phi_1, A_1), u_2 = (\phi_2, A_2) \in Y$, define the distance by

$$\rho_2(u_1, u_2) := \sup_{(x,y)\in X\times X} \|\phi_1(x,y) - \phi_2(x,y)\| + h(A_1, A_2),$$

where h is the Hausdorff metric on X.

First, we cite the following lemma which will be used later.

Lemma 4.1 (See [16]) *Let A and A_n $(n = 1, 2, \ldots)$ all be nonempty compact subsets of metric space X with $A_n \to A$ in Hausdorff metric topology, then the following statements hold*:

(i) $\bigcup_{n=1}^{+\infty} A_n \cup A$ *is also nonempty compact subset of* X.
(ii) *If* $x_n \in A_n$, $x_n \to x$, *then* $x \in A$.

Lemma 4.2 (Y, ρ_2) *is a complete metric space.*

Proof Clearly, ρ_2 is a metric on Y. We only need to show that (Y, ρ_2) is complete. Let $\{u_n = (\phi_n, A_n)\}$ be a Cauchy sequence of Y, then for any $\epsilon > 0$, there exists a positive integer $N(\epsilon)$ such that for any $m, n \geq N(\epsilon)$,

$$\rho_2(u_m, u_n) = \sup_{(x,y)\in X\times X} \|\phi_m(x,y) - \phi_n(x,y)\| + h(A_m, A_n) < \epsilon.$$

Hence, for any $m, n \geq N(\epsilon)$, there are

$$\sup_{(x,y)\in X\times X} \|\phi_m(x,y) - \phi_n(x,y)\| < \epsilon \quad \text{and} \quad h(A_m, A_n) < \epsilon.$$

Since H is a Banach space, for each $x, y \in X$, there exists $\phi(x,y) \in H$ such that $\lim_{m\to\infty} \phi_m(x,y) = \phi(x,y)$ and for any $n \geq N(\epsilon)$,

$$\sup_{(x,y)\in X\times X} \|\phi_n(x,y) - \phi(x,y)\| \leq \epsilon. \tag{20}$$

Since X is complete, $K(X)$ is also complete, where $K(X)$ denotes the space of all nonempty compact subsets of X and is endowed with the Hausdorff metric h induced by the metric on X. Consequently, by $h(A_m, A_n) < \epsilon$, there exists $A \in K(X)$ such that $A_n \to A$. Next, we will prove that $u := (\phi, A) \in Y$.

By the proof same as in Lemma 3.1, one can prove that for any $x, y \in X, x \neq y$, it holds that $\phi(x, y) + \phi(y, x) \notin \operatorname{int} C$ and $\sup_{(x,y)\in X\times X} \|\phi(x, y)\| < +\infty$.

Now we show that ϕ is C-upper semicontinuous on $X \times X$.

For any $x, y \in X$, let $\{x_n\}, \{y_n\} \subset X$ be such that $x_n \to x$, $y_n \to y$. Fix $n_0 \geq N(\epsilon)$, then, by (20),

$$\phi(x_n, y_n) - \phi_{n_0}(x_n, y_n) \in B(\epsilon) \tag{21}$$

and

$$\phi_{n_0}(x, y) - \phi(x, y) \in B(\epsilon). \tag{22}$$

Since ϕ_{n_0} is upper semicontinuous on $X \times X$ and $x_n \to x$, $y_n \to y$, there exists $N_1 > 0$ such that for any $n \geq N_1$, it holds that

$$\phi_{n_0}(x_n, y_n) - \phi_{n_0}(x, y) \in B^\circ(\epsilon) - C. \tag{23}$$

By (21)–(23), we obtain that for any $n \geq N_1$,

$$\begin{aligned}
\phi(x_n, y_n) &= \phi(x, y) + \big[\phi(x_n, y_n) - \phi_{n_0}(x_n, y_n)\big] + \big[\phi_{n_0}(x_n, y_n) - \phi_{n_0}(x, y)\big] \\
&\quad + \big[\phi_{n_0}(x, y) - \phi(x, y)\big] \\
&\in \phi(x, y) + B(\epsilon) + B^\circ(\epsilon) - C + B(\epsilon) \\
&= \phi(x, y) + B^\circ(3\epsilon) - C.
\end{aligned}$$

Thus, we obtain that ϕ is C-upper semicontinuous on $X \times X$.

Next we only need to prove that there exists $x^* \in A$ such that $\phi(x^*, y) \in C$ for all $y \in A$. For each $n = 1, 2, \ldots$, since $u_n = (\phi_n, A_n) \in Y$, there exists $x'_n \in A_n$ such that

$$\phi_n\big(x'_n, y\big) \in C, \quad \forall y \in A_n. \tag{24}$$

Since A and A_n $(n = 1, 2, \ldots)$ are all compact and $A_n \to A$, by Lemma 4.1(i), $\bigcup_{n=1}^{+\infty} A_n \cup A$ is also compact. Note that $\{x'_n\} \subset \bigcup_{n=1}^{+\infty} A_n \cup A$. Without loss of generality, we assume that $x'_n \to x^*$. Then by Lemma 4.1(ii), $x^* \in A$. For any $y \in A$, by virtue of $A_n \to A$, there exists $y'_n \in A_n$ (for each $n = 1, 2, \ldots$) such that $y'_n \to y$. By (24), it follows from $y'_n \in A_n$ that

$$\phi_n\big(x'_n, y'_n\big) \in C. \tag{25}$$

Since ϕ is C-upper semicontinuous on $X \times X$ and $x'_n \to x^*$, $y'_n \to y$, there exists $N_2 \geq N(\epsilon)$ such that for any $n \geq N_2$,

$$\phi\big(x^*, y\big) - \phi\big(x'_n, y'_n\big) \in B^\circ(\epsilon) + C. \tag{26}$$

From (20), we derive that for any $n \geq N_2$,

$$\phi\big(x'_n, y'_n\big) - \phi_n\big(x'_n, y'_n\big) \in B(\epsilon). \tag{27}$$

By virtue of (25)–(27), we obtain that

$$\begin{aligned}\phi(x^*, y) &= \big[\phi(x^*, y) - \phi(x'_n, y'_n)\big] + \big[\phi(x'_n, y'_n) - \phi_n(x'_n, y'_n)\big] + \phi_n(x'_n, y'_n) \\ &\in B^\circ(\epsilon) + C + B(\epsilon) + C \\ &\subset B^\circ(2\epsilon) + C. \end{aligned} \tag{28}$$

By the arbitrariness of $\epsilon > 0$, we derive from (28) that $\phi(x^*, y) \in C$ for all $y \in A$.

Thus, we have shown $u = (\phi, A) \in Y$. Consequently, inequality (20) and $A_n \to A$ imply that $\lim_{n\to\infty} \rho_2(u_n, u) = 0$. Therefore, (Y, ρ_2) is a complete metric space. □

For each $u = (\phi, A) \in Y$, by the definition of Y, the vector Ky Fan inequality ϕ must have at least one solution in A, i.e., $\exists x^* \in A$ such that $\phi(x^*, y) \in C$ for all $y \in A$. Denote by $S_2(u)$ the set of all solutions of the vector Ky Fan inequality ϕ in A. Then the correspondence $u \to S_2(u)$ defines a set-valued mapping $S_2 : Y \rightrightarrows X$.

In order to research the property of the mapping S_2, we give the following lemma.

Lemma 4.3 *If the vector-valued function $f : X \to H$ is C-upper semicontinuous on X, then the set $L := \{x \in X : f(x) \in C\}$ is closed in X.*

Proof Let $\{x_n\} \subset L$ with $x_n \to x \in X$. We only need to prove $x \in L$. It follows from $x_n \in L$ that $f(x_n) \in C$. Since $f : X \to H$ is C-upper semicontinuous at x and $x_n \to x$, there exists $N > 0$ such that for any $n > N$, it holds that

$$f(x) \in f(x_n) + B^\circ(\epsilon) + C \subset C + B^\circ(\epsilon) + C \subset B^\circ(\epsilon) + C.$$

By the arbitrariness of $\epsilon > 0$, we know that $f(x) \in C$, consequently $x \in L$. The proof is thus complete. □

Lemma 4.4 *$S_2 : Y \rightrightarrows X$ is an usco mapping.*

Proof For each $u = (\phi, A) \in Y$, note that

$$S_2(u) = \big\{x \in A : \phi(x, y) \in C,\ \forall y \in A\big\} = \bigcap_{y\in A} \big\{x \in A : \phi(x, y) \in C\big\}.$$

Since ϕ is C-upper semicontinuous on $X \times X$, it is also C-upper semicontinuous on $A \times A$. Consequently, $x \to \phi(x, y)$ is also C-upper semicontinuous on A. By Lemma 4.3, for each $y \in A$, the set $\{x \in A : \phi(x, y) \in C\}$ is closed in A. Thus, $S_2(u)$ is closed in A. Moreover, $S_2(u)$ is compact since A is compact.

Next, we prove that S_2 is upper semicontinuous on Y. Suppose, by contradiction, that there exists $u = (\phi, A) \in Y$ such that S_2 is not upper semicontinuous at u. Then there exists an open neighborhood G in X with $G \supset S_2(u)$ such that for each $n = 1, 2, \ldots$ and each open neighborhood $U_n := \{u' = (\phi', A') \in Y : \rho_2(u', u) < \frac{1}{n}\}$ of u, there exist $u_n = (\phi_n, A_n) \in U_n$ and $x_n \in S_2(u_n)$ but $x_n \notin G$.

Since $u_n = (\phi_n, A_n) \in U_n$ for each $n = 1, 2, \ldots$, we have $\rho_2(u_n, u) < \frac{1}{n} \to 0$. Then

$$\phi_n \to \phi \quad \text{and} \quad A_n \to A.$$

It follows from $x_n \in S_2(u_n)$ that $x_n \in A_n$ and

$$\phi_n(x_n, y) \in C, \quad \forall y \in A_n. \tag{29}$$

By Lemma 4.1(i), $\bigcup_{n=1}^{+\infty} A_n \cup A$ is compact due to the compactness of A_n and A. Note that $\{x_n\}_{n=1}^{+\infty} \subset \bigcup_{n=1}^{+\infty} A_n \cup A$. Without loss of generality, we suppose that $\{x_n\}_{n=1}^{+\infty}$ is convergent. Moreover, by Lemma 4.1(ii), the limit x^* of $\{x_n\}_{n=1}^{+\infty}$ belongs to A, i.e., $x_n \to x^* \in A$. Meanwhile, $x_n \notin G$ and G is open, thus $x^* \notin G$. Since $S_2(u) \subset G$, it holds that $x^* \notin S_2(u)$. Consequently, there exists $y \in A$ such that

$$\phi\bigl(x^*, y\bigr) \notin C. \tag{30}$$

Since $A_n \to A$ and $y \in A$, there exists a sequence $\{y_n\}_{n=1}^{+\infty}$ such that $y_n \in A_n$ and $y_n \to y$. Since $\phi_n \to \phi$, there exists $N_1 > 0$ such that for any $n \geq N_1$,

$$\phi(x_n, y_n) - \phi_n(x_n, y_n) \in B^\circ\left(\frac{\epsilon}{2}\right). \tag{31}$$

Moreover, ϕ is C-upper semicontinuous on $X \times X$ as well as $x_n \to x^*$, $y_n \to y$, hence there exists $N_2 > N_1$ such that for any $n \geq N_2$,

$$\phi\bigl(x^*, y\bigr) - \phi(x_n, y_n) \in B^\circ\left(\frac{\epsilon}{2}\right) + C. \tag{32}$$

By (29), (31), and (32), we have that for any $n \geq N_2$,

$$\begin{aligned}\phi\bigl(x^*, y\bigr) &= \bigl[\phi\bigl(x^*, y\bigr) - \phi(x_n, y_n)\bigr] + \bigl[\phi(x_n, y_n) - \phi_n(x_n, y_n)\bigr] + \phi_n(x_n, y_n)\\ &\in B^\circ\left(\frac{\epsilon}{2}\right) + C + B^\circ\left(\frac{\epsilon}{2}\right) + C\\ &\subset B^\circ(\epsilon) + C.\end{aligned}$$

By the arbitrariness of $\epsilon > 0$, we obtain that $\phi(x^*, y) \in C$, which is in contradiction with (30). Therefore, S_2 must be upper semicontinuous on Y. The proof is thus finished. □

Theorem 4.1 *There exists a dense residual subset Q_2 of Y such that $S_2(u)$ is a singleton for each $u \in Q_2$.*

Proof Y is a complete metric space, so it is a Baire space. By Lemma 4.4, $S_2 : Y \rightrightarrows X$ is an usco mapping. And X is a subspace of the Banach space E. Hence, by Theorem 2.2, there exists a dense residual subset Q_2 of Y such that S_2 is lower semicontinuous at each $u = (\phi, A) \in Q_2$.

Suppose, by contradiction, that $S_2(u_0)$ is not a singleton for some $u_0 = (\phi_0, A_0) \in Q_2$. Then there exist at least two points $x_1, x_2 \in S_2(u_0) \subset A_0$ with $x_1 \neq x_2$. Therefore, there exists an open neighborhood U of x_1 and an open neighborhood V of x_2 such that $U \cap V = \emptyset$.

Define a function $g : X \to \mathbb{R}$ by

$$g(x) := \frac{d(x, x_1)}{d(x, x_1) + d(x, X \setminus U)}, \quad \forall x \in X.$$

where d is the metric on X. Note that $g : X \to \mathbb{R}$ is continuous on X; $\forall x \in X$, $0 \leq g(x) \leq 1$; $g(x_1) = 0$; $\forall x \in V$, $g(x) = 1$.

Take $z \in \operatorname{int} C$. For each $n = 1, 2, \dots$, let $\phi_n : X \times X \to H$ be defined by

$$\phi_n(x, y) := \phi_0(x, y) + \frac{1}{n}\big[g(y) - g(x)\big]z, \quad \forall x, y \in X,$$

and u_n be defined by

$$u_n := (\phi_n, A_0).$$

By the processes similar to that of Theorem 3.1, we can check that $u_n \in Y$ for each $n = 1, 2, \dots$ and $u_n \to u_0$ as $n \to \infty$.

Note that $x_2 \in V \cap S_2(u_0)$, then $V \cap S_2(u_0) \neq \emptyset$. Since S_2 is lower semicontinuous at u_0 and $u_n \to u_0$, there exists $n_0 > 0$ big enough such that $V \cap S_2(u_{n_0}) \neq \emptyset$. Take $x_{n_0} \in V \cap S_2(u_{n_0})$, then

$$x_{n_0} \in V \cap A_0 \quad \text{and} \quad \phi_{n_0}(x_{n_0}, y) \in C, \quad \forall y \in A_0.$$

Taking $y = x_1 (\in A_0)$, we obtain that

$$\phi_{n_0}(x_{n_0}, x_1) \in C. \tag{33}$$

Since $g(x_1) = 0$ and $g(x_{n_0}) = 1$ (because of $x_{n_0} \in V$), it follows that

$$\phi_{n_0}(x_{n_0}, x_1) = \phi_0(x_{n_0}, x_1) + \frac{1}{n_0}\big[g(x_1) - g(x_{n_0})\big]z = \phi_0(x_{n_0}, x_1) - \frac{1}{n_0}z. \tag{34}$$

Noting that $\frac{1}{n_0}z \in \operatorname{int} C$, as well as (33), (34), and Lemma 2.2, we get

$$\phi_0(x_{n_0}, x_1) = \phi_{n_0}(x_{n_0}, x_1) + \frac{1}{n_0}z \in C + \operatorname{int} C \subset \operatorname{int} C. \tag{35}$$

Since $x_1 \in S_2(u_0)$, we have $\phi_0(x_1, y) \in C$ for all $y \in A_0$. Especially, taking $y = x_{n_0} (\in A_0)$, we get

$$\phi_0(x_1, x_{n_0}) \in C. \tag{36}$$

It follows from (35), (36), and Lemma 2.2 that

$$\phi_0(x_{n_0}, x_1) + \phi_0(x_1, x_{n_0}) \in \operatorname{int} C + C \subset \operatorname{int} C. \tag{37}$$

But $u_0 = (\phi_0, A_0) \in Y$ and $x_1 \neq x_{n_0}$ imply that

$$\phi_0(x_{n_0}, x_1) + \phi_0(x_1, x_{n_0}) \notin \operatorname{int} C,$$

which is in contradiction with (37). Therefore, $S_1(u_0)$ must be a singleton. □

When $H = \mathbb{R}$, $C = [0, +\infty[$, we get the following Corollary 4.1.

Corollary 4.1 *Let*

$$Y' := \left\{ u = (f, A) : \begin{array}{l} f : X \times X \to \mathbb{R} \textit{ is upper semicontinuous on } X \times X, \\ \forall x, y \in X,\ x \neq y,\ f(x, y) + f(y, x) \leq 0, \\ \sup\limits_{(x,y) \in X \times X} \big|f(x, y)\big| < +\infty, \\ A \textit{ is a nonempty compact subset of } X \textit{ and} \\ \exists x \in A \textit{ such that } f(x, y) \geq 0,\ \forall y \in A \end{array} \right\}.$$

Then there exists a dense residual subset Q'_2 of Y' such that for each $u=(f,A)\in Q'_2$, f has a unique equilibrium point in A.

Remark 4.1 Similarly to Remark 3.1, Corollary 4.1 generalized Theorem 3.1 of [12], another main result of [12], from four aspects.

5 Remarks

Let us give several remarks on our results through two examples.

Remark 5.1 We emphasize that M and Y must be nonempty, which can be seen from two very simple examples as follows.

Example 5.1 Let X be a nonempty compact subset of a metric space and $\phi(x,y)\equiv 0$ for any $x,y\in X$, where 0 is the zero element of a Banach space H and C is a nonempty, closed, convex, and pointed cone in H with apex at 0 and $\operatorname{int} C\neq\emptyset$. One can easily check that $\phi\in M$.

Example 5.2 Let X be a nonempty, bounded, and closed subset of a complete metric space, $\phi(x,y)\equiv 0$ for any $x,y\in X$, $A\in K(X)$ and $u=(\phi,A)$, where 0 is the zero element of H, $K(X)$ is the space of all nonempty compact subsets of X and C is a nonempty, closed, convex, and pointed cone in H with apex at 0 and $\operatorname{int} C\neq\emptyset$. One can also easily check that $u\in Y$.

Remark 5.2 One can not conclude that $S_1(\phi)$ or $S_2(u)$ is a singleton for each $\phi\in M$ or each $u\in Y$. In fact, $S_1(\phi)=X$ for Example 5.1 and $S_2(u)=A$ for Example 5.2, that is, neither $S_1(\phi)$ nor $S_2(u)$ is a singleton, which implies that the dense residual subsets $Q_1\neq M$, $Q_2\neq Y$.

Remark 5.3 Though not all the vector Ky Fan inequalities in M (or Y) have unique solution, Theorem 3.1 (or Theorem 4.1) tells us that, in the sense of Baire category, most of the vector Ky Fan inequalities in M (or Y) have unique solution. Moreover, since Q_1 (or Q_2) is everywhere dense in M (or Y), every vector Ky Fan inequality in M (or Y) can be approached arbitrarily by a sequence of vector Ky Fan inequalities in Q_1 (or Q_2) each of which has a unique solution.

6 Conclusions

According two different perturbations, we propose two complete metric spaces consisting of vector Ky Fan inequalities. In both spaces, we all derive the generic uniqueness of solutions.

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